Density of positive closed currents and dynamics of Hénon-type automorphisms of $\mathbb{C}^k$ (part I)

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March 1, 2013

Abstract

We introduce a new method to prove equidistribution properties in complex dynamics of several variables. We obtain the equidistribution for saddle periodic points of Hénon-type maps on $\mathbb{C}^k$. A key point of the method is a notion of density which extends both the notion of Lelong number and the theory of intersection for positive closed currents on Kähler manifolds. Basic calculus on the density of currents is established.

Classification AMS 2010: 32U, 37F, 32H50
Keywords: density of currents, tangent current, intersection of currents, Hénon-type map, Green current, laminar current, Green measure, saddle periodic point.

1 Introduction

The first aim of this work is to introduce a new approach in order to get several equidistribution properties in complex dynamics in higher dimension. The strategy that we will describe in the case of Hénon-type automorphisms, requires developments of the theory of positive closed currents which are of independent interest.

Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a polynomial automorphism. We extend it to a birational self-map of the projective space $\mathbb{P}^k$ that we still denote by f. We assume that f is not an automorphism of $\mathbb{P}^k$; otherwise the associated dynamical system is elementary. We say that f is *regular or of Hénon-type* if the indeterminacy sets I_+ and I_- of f and of its inverse f^{-1} satisfy $I_+ \cap I_- = 0$. We refer to [35] for basic properties of this class of maps. In dimension 2, Hénon maps satisfy this property and any dynamically interesting polynomial automorphism is conjugated to a Hénon map, see Friedland-Milnor [27].

Consider a Hénon-type map f as above. There is an integer $1 \leq p \leq k-1$ such that $\dim I_+ = k-p-1$ and $\dim I_- = p-1$. Let d_+ (resp. d_-) denote the algebraic degrees of f^+ (resp. of f^-), i.e. the maximal degrees of its components

which are polynomials in $\mathbb{C}^k$. It follows that $d_+^p = d_-^{k-p}$, we denote this integer by d. In [35], the second author constructed for such a map an invariant measure μ with compact support in $\mathbb{C}^k$ which turns out to be the unique measure of maximal entropy $\log d$, see de Thélin [12].

The measure μ is called *the Green measure or the equilibrium measure of f*. It is obtained as the intersection of the main Green current T_+ of f and the one associated to f^{-1}. The authors have shown that T_+ (resp. T_-) is the unique positive closed (p,p)-current (resp. $(k-p,k-p)$-current) of mass 1 supported by the set $\mathscr{K}_+$ (resp. $\mathscr{K}_-$) of points of bounded orbit (resp. backward orbit) in $\mathbb{C}^k$. They are also the unique currents having no mass at infinity which are invariant under $d^{-1}f^*$ (resp. $d^{-1}f_*$), see [20].

Let P_n denote the set of periodic points of period n of f in $\mathbb{C}^k$ and SP_n the set of saddle periodic points of period n in $\mathbb{C}^k$. We have the following result.

Theorem 1.1. *Let f, d, μ, P_n and SP_n be as above. Then the saddle periodic points of f are asymptotically equidistributed with respect to μ. More precisely, if Q_n denotes P_n or SP_n we have*

$$d^{-n} \sum_{a \in Q_n} \delta_a \to \mu \qquad \textit{as} \quad n \to \infty,$$

where δ_a denotes the Dirac mass at a.

We can replace Q_n with other subsets of SP_n. This gives the nature of typical periodic points. For example, given a small number $\epsilon > 0$, we can take only periodic points a of period n such that the differential Df^n at a admits p eigenvalues of modulus larger than $(\delta - \epsilon)^{n/2}$ and $k-p$ eigenvalues of modulus smaller than $(\delta - \epsilon)^{-n/2}$ with $\delta := \min(d_+, d_-)$. Note that an arithmetic version of the above theorem is obtained independently by Lee in [33]. The result in the present paper was announced in the Shirahama conference in December 2011.

In dimension 2, Theorem 1.1 and the uniqueness of the maximal entropy measure were obtained by Bedford-Lyubich-Smillie [1, 2] and the uniqueness of $T_\pm$ were obtained by Fornæss and the second author [25]. In order to obtain the equidistribution of periodic points in dimension 2, Bedford-Lyubich-Smillie proved and used that the Green currents T_+, T_- are laminated by Riemann surfaces whose intersections give the measure μ. Their approach uses heavily the fact that these currents are of (complex) dimension and of codimension 1, see also Dujardin [23] for a more systematic treatment of that approach.

In the higher dimensional case, we will use another method which also allows us to obtain as a consequence the laminar property of $T_\pm$ (a weaker result has been obtained by the first author in [14]) and the product structure of μ. The approach, that we describe below, has some advantages. It permits to show in the same way other equidistribution properties, for example, if L_+ and L_- are Zariski generic subvarieties of dimension $k-p$ and p respectively, then the points

in $f^{-n}(L_+) \cap f^n(L_-)$ are also equidistributed with respect to μ when n goes to infinity. One can hope that our approach will allow to estimate the speed of convergence in the above equidistribution results.

Let Δ denote the diagonal of $\mathbb{P}^k \times \mathbb{P}^k$ and Γ_n denote the compactification of the graph of f^n in $\mathbb{P}^k \times \mathbb{P}^k$. The set P_n can be identified with the intersection of Γ_n and Δ in $\mathbb{C}^k \times \mathbb{C}^k$. The dynamical system associated to the map $F := (f, f^{-1})$ on $\mathbb{P}^k \times \mathbb{P}^k$ is similar to the one associated to Hénon-type maps on $\mathbb{P}^k$. It was used by the first author in [15] in order to obtain the exponential mixing of μ on $\mathbb{C}^k$. Observe that Γ_n is the pull-back of Δ or Γ_1 by $F^{n/2}$ or $F^{(n-1)/2}$. So a property similar to the uniqueness of the main Green currents mentioned above implies that the positive closed (k,k)-current $d^{-n}[\Gamma_n]$ converges to the main Green current of F which is equal to $T_+ \otimes T_-$. Therefore, since $\mu = T_+ \wedge T_-$ can be identified with $[\Delta] \wedge (T_+ \otimes T_-)$, Theorem 1.1 is equivalent to

$$\lim_{n\to\infty} [\Delta] \wedge d^{-n}[\Gamma_n] = [\Delta] \wedge \lim_{n\to\infty} d^{-n}[\Gamma_n]$$

on $\mathbb{C}^k \times \mathbb{C}^k$. So our result requires the development of a good intersection theory in any dimension.

The typical difficulty is illustrated in the following example. Consider Δ' the unit disc in $\mathbb{C} \times \{0\} \subset \mathbb{C}^2$ and Γ'_n the graph of the function $x \mapsto x^{d^n}$ over Δ'. The currents $d^{-n}[\Gamma'_n]$ converge to a current on the boundary of the unit bidisc in $\mathbb{C}^2$ while their intersection with $[\Delta']$ is the Dirac mass at 0. So we have

$$\lim_{n\to\infty} [\Delta'] \wedge d^{-n}[\Gamma'_n] \neq [\Delta'] \wedge \lim_{n\to\infty} d^{-n}[\Gamma'_n]$$

We see in this example that Γ'_n is tangent to Δ' at 0 with maximal order. We can perturb Γ'_n in order to get manifolds which intersect Δ' transversally but the limit of their intersections with Δ' is still equal to the Dirac mass at 0. In fact, this phenomenon is due to the property that some tangent lines to Γ'_n are too close to tangent lines to Δ'.

It is not difficult to construct a map f such that Γ_n is tangent or almost tangent to Δ at some points for every n. In order to handle the main difficulty in our problem, the strategy is to show that the almost tangencies become negligible when n tends to infinity. This property is translated in our study into the fact that a suitable density for positive closed currents vanishes. Then, a geometric approach developed in [14] allows us to obtain the result. We will give the details in the second part of this article. We explain now the notion of density of currents in the dynamical setting and then develop the theory in the general setting of arbitrary positive closed currents.

Let $\mathrm{Gr}_k(\mathbb{P}^k \times \mathbb{P}^k)$ denote the Grassmannian bundle over $\mathbb{P}^k \times \mathbb{P}^k$ where each point corresponds to a pair $(x, [v])$ of a point $x \in \mathbb{P}^k \times \mathbb{P}^k$ and of the direction $[v]$ of a simple k-vector v in the complex tangent space to $\mathbb{P}^k \times \mathbb{P}^k$ at x. Let $\widehat{\Gamma}_n$ denote the lift of Γ_n to $\mathrm{Gr}_k(\mathbb{P}^k \times \mathbb{P}^k)$, i.e. the set of points $(x, [v])$ with $x \in \Gamma_n$

and v a k-vector tangent to Γ_n at x. Let $\widetilde{\Delta}$ denote the set of points $(x,[v])$ in $\mathrm{Gr}_k(\mathbb{P}^k \times \mathbb{P}^k)$ with $x \in \Delta$ and v non-transverse to Δ. The intersection $\widehat{\Gamma}_n \cap \widetilde{\Delta}$ corresponds to the non-transverse points of intersection between Γ_n and Δ. Note that $\dim \widehat{\Gamma}_n + \dim \widetilde{\Delta}$ is smaller than the dimension of $\mathrm{Gr}_k(\mathbb{P}^k \times \mathbb{P}^k)$ and the intersection of subvarieties of such dimensions are generically empty. Analogous construction can be done for the manifolds Γ'_n and Δ' given above.

We show that the currents $d^{-n}[\widehat{\Gamma}_n]$ converge to some positive closed current $\widehat{\mathbb{T}}$ which is considered as the lift of $\mathbb{T} := T_+ \otimes T_-$ to $\mathrm{Gr}_k(\mathbb{P}^k \times \mathbb{P}^k)$. Using a theorem due to de Thélin [13] on the hyperbolicity of μ we show that the density between $\widehat{\mathbb{T}}$ and $\widetilde{\Delta}$ vanishes. This property says that almost tangencies are negligible when n goes to infinity. The above example with Γ'_n and Δ' is an illustration of the opposite situation.

Consider now the general situation on a Kähler manifold X of dimension k. Assume for simplicity that X is compact. We want to introduce a notion of density between two positive closed currents T_1 and T_2 on X of bidegrees (p_1,p_1) and (p_2,p_2) respectively. Consider first the case where T_1 and T_2 are given by integration on submanifolds V_1 and V_2 such that $\dim V_1 + \dim V_2 < k$. For generic such submanifolds, we have $V_1 \cap V_2 = \varnothing$. However, in general this intersection may be non-empty and the classical intersection theory of currents does not give a meaning to this intersection for bi-degree reason.

On the other hand, when V_2 is a point, denoted by a, there is a notion of multiplicity of V_1 at a. More generally, if T_1 is a general positive closed current there is a notion of Lelong number $\nu(T_1,a)$ of T_1 at a which represents the density of T_1 at a. We first recall this notion and then extend it to the general case. For more detailed expositions on Lelong numbers, see Demailly [10], Lelong [34] and Siu [37].

Choose a local holomorphic coordinate system x near a such that $a = 0$ in these coordinates. The Lelong number of T_1 at a is the limit of the normalized mass of T_1 on the ball $\mathbb{B}(0,r)$ of center 0 and of radius r when r tends to 0. More precisely, we have

$$\nu(T_1,a) := \lim_{r\to 0} \frac{\|T_1\|_{\mathbb{B}(0,r)}}{(2\pi)^{k-p_1} r^{2k-2p_1}}\cdot$$

Note that $(2\pi)^{k-p_1} r^{2k-2p_1}$ is the mass on $\mathbb{B}(0,r)$ of the (p_1,p_1)-current of integration on a linear subspace through 0. Lelong proved that this limit always exists [34]. Thie showed that when T_1 is given by an analytic set this number is the multiplicity of V_1 at a. Siu proved that the limit does not depend on the choice of coordinates and that the function $a \mapsto \nu(T_1,a)$ is upper semi-continuous for the Zariski topology [37].

Let $\sigma : \widehat{X} \to X$ be the blow-up of X at a. The pull-back of T_1 to $\widehat{X} \setminus \sigma^{-1}(a)$ is a positive closed current that can be extended by 0 through the exceptional hypersurface $\sigma^{-1}(a) \simeq \mathbb{P}^{k-1}$. We call it the strict transform of T_1 and denote it by $\sigma^\diamond(T_1)$. In general the class of this current in the de Rham cohomology $H^*(\widehat{X},\mathbb{C})$

is not equal to the pull-back by σ of the class of T_1 in $H^*(X,\mathbb{C})$. According to Siu's results [37], the missing class can be represented by $\nu(T_1,a)$ times the class of a linear subspace in $\sigma^{-1}(a)$.

We can also consider the Lelong number from another geometric point of view related to Harvey's exposition [29]. Let $A_\lambda:\mathbb{C}^k\to\mathbb{C}^k$ be defined by $A_\lambda(x):=\lambda x$ with $\lambda\in\mathbb{C}^*$. When λ goes to infinity, the domain of definition of the current $T_{1,\lambda}:=(A_\lambda)_*(T_1)$ converges to $\mathbb{C}^k$. This family of currents is relatively compact and any limit current, for $\lambda\to\infty$, is invariant under the action of $\mathbb{C}^*$, i.e. invariant under $(A_\lambda)_*$. If S is a limit current, we can extend it to $\mathbb{P}^k$ with zero mass on the hyperplane at infinity. Thus, there is a positive closed current S_∞ on $\mathbb{P}^{k-1}$ such that $S=\pi_\infty^*(S_\infty)$. Here we identify the hyperplane at infinity with $\mathbb{P}^{k-1}$ and we denote by $\pi_\infty:\mathbb{P}^k\setminus\{0\}\to\mathbb{P}^{k-1}$ the canonical central projection (we do not consider the case where T_1 is a measure, i.e. $p_1=k$). The class of S_∞ (resp. of S) in the de Rham cohomology of $\mathbb{P}^{k-1}$ (resp. of $\mathbb{P}^k$) is equal to $\nu(T_1,a)$ times the class of a linear subspace. So these cohomology classes do not depend on the choice of S. Kiselman showed that in general the current S is not unique [32]. Blel-Demailly-Mouzali gave in [5] conditions on T_1 for the uniqueness of S.

We consider now the situation where T_1 is a general positive closed (p_1,p_1)-current and T_2 is given by a submanifold V_2. For simplicity, we will write T,p,V instead of T_1,p_1,V_2 and denote by l the dimension of V. With respect to the above case, the point a is replaced by the manifold V. We want to define a notion of tangent current to T along V that corresponds to the currents S above. Let E denote the normal vector bundle to V in X and $\overline{E}$ its canonical compactification. Denote by $A_\lambda:\overline{E}\to\overline{E}$ the map induced by the multiplication by λ on fibers of E with $\lambda\in\mathbb{C}^*$. We identify V with the zero section of E. The tangent currents to T along V will be positive closed (p,p)-currents on $\overline{E}$ which are V-conic, i.e. invariant under the action of A_λ. The first difficulty is that when V has positive dimension, in general, no neighbourhood of V in X is biholomorphic to a neighbourhood of V in E.

Let τ be a diffeomorphism between a neighbourhood of V in X and a neighbourhood of V in E whose restriction to V is identity. We assume that τ is admissible in the sense that the endomorphism of E induced by the differential of τ is identity. It is not difficult to show that such maps exist, see Lemma 4.2. Here is the main result in the first part of this paper. It is a consequence of Proposition 4.4 and Theorem 4.6 below.

Theorem 1.2. *Let $X,V,T,E,\overline{E},A_\lambda$ and τ be as above. Then the family of currents $T_\lambda:=(A_\lambda)_*\tau_*(T)$ is relatively compact and any limit current, for $\lambda\to\infty$, is a positive closed (p,p)-current on E whose trivial extension is a positive closed (p,p)-current on $\overline{E}$. Moreover, if S is such a current, it is V-conic, i.e. invariant under $(A_\lambda)_*$, and its de Rham cohomology class in $H^{2p}(\overline{E},\mathbb{C})$ does not depend on the choice of τ and of S.*

We say that S is a tangent current to T along V. Its class in the de Rham

cohomology group is the total tangent class to T along V. Note that this notion generalizes a notion of tangent cone in the algebraic setting where T is also given by a manifold, see Fulton [28] for details. The key point in the dynamical setting considered above is that the tangent currents to $\widehat{\mathbb{T}}$ along $\widetilde{\Delta}$ vanish.

The cohomology ring of $\overline{E}$ is generated by the cohomology ring of V and the tautological $(1,1)$-class on $\overline{E}$. Therefore, we can decompose the class of S and associate to it classes of different degrees on V. These classes represent different parts of the tangent class of T along V.

Consider now arbitrary positive closed currents T_1, T_2 on X and the tensor product $T_1 \otimes T_2$ on $X \times X$. When T_1, T_2 are currents of integration on manifolds V_1 and V_2, the tensor product $T_1 \otimes T_2$ is the current of integration on $V_1 \times V_2$. Let Δ denote the diagonal of $X \times X$. We can consider the tangent currents and the total tangent class to $T_1 \otimes T_2$ along Δ. The normal vector bundle to Δ is canonically isomorphic to the tangent bundle of X if we identify Δ with X. The tangent currents and the total tangent class in this case induce the density currents and the total density class associated to T_1 and T_2.

Assume that $p_1 + p_2 \leq k$ and that there is a only one tangent current S to $T_1 \otimes T_2$ along Δ. Assume also that for $j > k - p_1 - p_2$, the current S vanishes on the pull-back of (j,j)-forms by the canonical projection onto X. Then we show that S is the pull-back of a unique positive closed current S^h of bidegree $(p_1 + p_2, p_1 + p_2)$ on X. In this case, we call S^h the wedge-product of T_1 and T_2 and denote it by $T_1 \curlywedge T_2$. The notion can be extended to a finite number of currents. So the density of currents extends the theory of intersection. This is the intersection theory we use in the dynamical context in order to prove that some densities vanish.

In Sections 2 and 3, we will recall some basic notions on positive closed currents and we give several properties that will be used in our study. In particular, we introduce the $\star$-norm for currents which is useful in the mass estimates of the currents T_λ in Theorem 1.2 and of wedge-products of currents. We also prove extension results needed when dealing with blow-up and with the map τ which is not holomorphic. We then introduce the notion of horizontal dimension of a current on a projective fibration and the notion of V-conic currents on a vector bundle over V.

Tangent currents and tangent classes will be introduced in Section 4. We will prove there a property of semi-continuity of the tangent class which is similar to the semi-continuity of Lelong number with respect to the current. We will also give several properties which allow to compute tangent classes. In particular, we can compute such classes using strict transforms of current and blow-up of manifolds as in Siu's results on Lelong numbers. Finally, the density of currents and the first properties of a new intersection theory are presented in Section 5. We will compare our definition with a classical notion of intersection of $(1,1)$-currents. Applications to dynamics will be given in the second part of the paper.

Notations. Through the paper, we denote by $\mathbb{D}^k$ the unit polydisc in $\mathbb{C}^k$ and $\lambda\mathbb{D}^k$ the polydisc of radius λ centered at the origin of $\mathbb{C}^k$.

If X is an oriented manifold, denote by $H^*(X,\mathbb{C})$ the de Rham cohomology group of X and $H^*_c(X,\mathbb{C})$ the de Rham cohomology group defined by forms or currents with compact support in X. If V is a submanifold of X, denote by $H^*_V(X,\mathbb{C})$ the de Rham cohomology group defined in the same way using forms or currents on X whose supports intersect V in a compact set.

If T is a closed current on X denote by $\{T\}$ its class in $H^*(X,\mathbb{C})$. When T is supposed to have compact support then $\{T\}$ denotes the class of T in $H^*_c(X,\mathbb{C})$. If we only assume that $\mathrm{supp}(T)\cap V$ is compact, then $\{T\}$ denotes the class of T in $H^*_V(X,\mathbb{C})$. The current of integration on an oriented submanifold Y is denoted by $[Y]$. Its class is denoted by $\{Y\}$.

The restriction to a submanifold V of smooth forms on X defines a canonical morphism from $H^*_V(X,\mathbb{C})$ to $H^*_c(V,\mathbb{C})$; the restriction to V of a class is denoted by $\{\cdot\}_{|V}$. Currents on V can be canonically sent by the embedding map to currents on X. This induces a natural morphism from $H^*_c(V,\mathbb{C})$ to $H^*_V(X,\mathbb{C})$. The composition of the above two morphisms is equal to the endomorphism of the space $\oplus H^*_V(X,\mathbb{C})$ induced by the cup-product with $\{V\}$.

The group $H^{2k}_c(X,\mathbb{C})$ of maximal degree is often identified with $\mathbb{C}$ via the integration of forms of maximal degree on X. If X is a compact Kähler manifold, the groups $H^*(X,\mathbb{C})$, $H^*_c(X,\mathbb{C})$ and $H^*_V(X,\mathbb{C})$ are equal and we identify $H^p(X,\mathbb{C})$ with the direct sum of the Hodge cohomology groups $H^{q,p-q}(X,\mathbb{C})$ via the Hodge decomposition.

2 Positive currents and spaces of test forms

In this section, we recall some basic notions on positive currents on a complex manifold and refer the reader to Demailly [10], de Rham [11], Federer [24], Hörmander [30, 31], Siu [37] and to [22] for details. We will also introduce and study some spaces of test forms which are the core of the technical part of our work. They will permit, in particular, to bound the mass of the currents T_λ in Theorem 1.2 and to show that their $(q,2p-q)$-components, with $q\neq p$, converge to 0 when λ tends to infinity.

Let X be a complex manifold of dimension k. A (p,p)-form θ on X is *positive* if for any point in X we can write θ as a finite combination of forms of type

$$(i\gamma_1\wedge\overline{\gamma}_1)\wedge\ldots\wedge(i\gamma_p\wedge\overline{\gamma}_p),$$

where $\gamma_1,\ldots,\gamma_p$ are $(1,0)$-forms. A (p,p)-current T on X is *weakly positive* if $T\wedge\theta$ is a positive measure for any smooth positive $(k-p,k-p)$-form θ. Such a current is of order 0 and real, i.e. $T=\overline{T}$.

A (p,p)-current T is *positive* if $T\wedge\theta$ is a positive measure for any smooth weakly positive $(k-p,k-p)$-form θ. Positive currents and positive forms are

weakly positive. Positivity and weak positivity are local properties. They coincide only for bidegree (p,p) with $p = 0, 1, k-1$ or k. On a chart of X, in the definition of (weakly) positive current, it suffices to use only forms θ with constant coefficients. Positive Hermitian $(1,1)$-forms on X are examples of positive forms. A (p,p)-current T is *strictly positive* if for a fixed smooth Hermitian $(1,1)$-form β on X we have locally $T \geq \epsilon\beta^p$, i.e. $T - \epsilon\beta^p$ is positive, for some constant $\epsilon > 0$. The definition does not depend on the choice of β.

From now on, consider a Kähler manifold X of dimension k, not necessarily compact. Let ω be a fixed Kähler form on X. It induces a Kähler metric on X and also metrics on the vector bundles of differential forms. This permits to define the mass-norm for currents of order 0 on X. If T is a current of order 0 and K a Borel subset of X, the mass of T on K is denoted by $\|T\|_K$ and the mass of T on X is denoted by $\|T\|$. If T is a (weakly) positive or negative (i.e. $-T$ is positive or weakly positive) (p,p)-current, the above mass-norm is equivalent to the mass of the trace measure $T \wedge \omega^{k-p}$. Then, we identify $\|T\|_K$ with the mass of $T \wedge \omega^{k-p}$ on K.

We introduce now some spaces of test forms and establish properties that we will use later to estimate the mass of currents. Fix open subsets W_1 and W_2 of X with smooth boundaries such that $W_1 \cap W_2$ is relatively compact in X. The notions below depend on the choice of W_1 and W_2.

Definition 2.1. Let R be a $(1,1)$-current of order 0 on X with no mass outside $W_1 \cap \overline{W}_2$. We define the $\star$-norm $\|R\|_\star$ of R as the infimum of the constants $c \geq 0$ such that the real and imaginary parts of R satisfy

$$-c(\omega + dd^c\phi) \leq \mathrm{Re}(R), \mathrm{Im}(R) \leq c(\omega + dd^c\phi)$$

for some quasi-psh function ϕ on W_1 satisfying $dd^c\phi \geq -\omega$ on W_1 and which vanishes outside W_2. By convention, if such constant does not exist, the $\star$-norm of R is infinite.

Note that when $\|R\|_\star$ is finite, R is absolutely continuous with respect to the positive closed $(1,1)$-current $R' := \omega + dd^c\phi$ on W_1. In particular, the trace measure $R \wedge \omega^{k-1}$ of R is equal to the product of a bounded function with the trace measure $R' \wedge \omega^{k-1}$ of R'. It is not difficult to check that $\|\cdot\|_\star$ defines a norm on the space of $(1,1)$-currents R with $\|R\|_\star$ finite. This space contains the $\mathscr{C}^2$ forms with support in $W_1 \cap W_2$. We will use it as a space of test forms in order to study currents with support in W_1.

Definition 2.2. Let Γ be a form of bidegree $(1,0)$ or $(0,1)$ vanishing outside $W_1 \cap \overline{W}_2$ with L^2_{loc} coefficients in $W_1 \cap \overline{W}_2$. We define the $\star$-norm of Γ by $\|\Gamma\|_\star := \|i\Gamma \wedge \overline{\Gamma}\|_\star^{1/2}$. If Γ is an L^2 1-form vanishing outside $W_1 \cap \overline{W}_2$, we define $\|\Gamma\|_\star$ as the supremum of the $\star$-norms of its bidegree $(1,0)$ and bidegree $(0,1)$ components. By convention, if Γ is a 1-current which is not given by an L^2_{loc} form on $W_1 \cap \overline{W}_2$, its $\star$-norm is infinite.

Remark 2.3. A version of the $\star$-norm was introduced and used by the authors for positive closed currents on compact Kähler manifolds [17, 20, 21], see also Vigny [39]. We can easily extend it to currents of bidegree (p,p), $(p,0)$ or $(0,p)$. For currents R of bidegree (p,q) we can consider the square root of the $\star$-norm of $\Gamma\otimes\overline{\Gamma}$ in $X\times X$. This quantity was implicitly used in some dynamical problems, see [14].

Lemma 2.4. *The map $\Gamma\mapsto\|\Gamma\|_\star$ defines a norm on the space of 1-forms Γ vanishing outside $W_1\cap\overline{W}_2$ such that $\|\Gamma\|_\star$ is finite. If Γ_1,Γ_2 are such forms then*

$$\|\Gamma_1\wedge\overline{\Gamma}_2\|_\star\leq\|\Gamma_1\|_\star\|\Gamma_2\|_\star.$$

Proof. For the first assertion, it is enough to prove the triangle inequality for the bidegree $(1,0)$-case. Let Γ_1,Γ_2 be of bidegree $(1,0)$ and define $c:=\|\Gamma_1\|_\star^{-1}\|\Gamma_2\|_\star$. Since the form $i(c\Gamma_1-c^{-1}\Gamma_2)\wedge(c\overline{\Gamma}_1-c^{-1}\overline{\Gamma}_2)$ is positive, we have

$$i(\Gamma_1+\Gamma_2)\wedge(\overline{\Gamma}_1+\overline{\Gamma}_2)\leq(1+c)(i\Gamma_1\wedge\overline{\Gamma}_1)+(1+c^{-1})(i\Gamma_2\wedge\overline{\Gamma}_2).$$

We then deduce without difficulty that $\|\Gamma_1+\Gamma_2\|_\star\leq\|\Gamma_1\|_\star+\|\Gamma_2\|_\star$.

For the second assertion, the positivity of the above form also implies

$$-2\mathrm{Im}(\Gamma_1\wedge\overline{\Gamma}_2)=i\Gamma_1\wedge\overline{\Gamma}_2+i\Gamma_2\wedge\overline{\Gamma}_1\leq c(i\Gamma_1\wedge\overline{\Gamma}_1)+c^{-1}(i\Gamma_2\wedge\overline{\Gamma}_2).$$

It follows that $\|\mathrm{Im}(\Gamma_1\wedge\overline{\Gamma}_2)\|_\star\leq\|\Gamma_1\|_\star\|\Gamma_2\|_\star$. We obtain a similar inequality for $\mathrm{Re}(\Gamma_1\wedge\overline{\Gamma}_2)$ by replacing Γ_1 with $i\Gamma_1$. This completes the proof of the lemma. □

Definition 2.5. A current R of order 0 on X is said to be *a quasi-continuous current or quasi-continuous form* if it vanishes outside some open set $W_R\Subset X$ and is given on that open set by a continuous form Θ which is also an L^1 form. The open set of points $x\in W_R$ such that $\Theta(x)\neq 0$ is called *an essential support* of R.

Note that the essential support of R depends on the choice of Θ and W_R; it is unique up to a set of zero Lebesgue measure. If T is a current of order 0 and if R is as above, we can define the current $T\wedge R$ on W_R. If this wedge-product has finite mass, we can extend it by 0 to a current on X that we still denote by $T\wedge R$. The wedge-products we will consider below satisfy this property. We will estimate it mass not in term of the L^∞ norm of R.

Lemma 2.6. *Let T be a positive closed (p,p)-current on X such that* $\mathrm{supp}(T)\subset W_1$. *Let R be a quasi-continuous form of bidegree $(1,1)$ vanishing outside $W_1\cap W_2$ and with finite $\star$-norm. Then there is a constant $c>0$ independent of T,R and a positive closed $(p+1,p+1)$-current T' on X such that*

$$-T'\leq\mathrm{Re}(T\wedge R),\mathrm{Im}(T\wedge R)\leq T'\quad\textit{and}\quad\|T'\|_W=c\|T\|_W\|R\|_\star$$

for any open set W which contains $\overline{W}_2$.

Proof. We can assume that R is a real current such that $\|R\|_\star = 1/2$ and that there is a quasi-psh function ϕ on W_1 which vanishes outside W_2 and satisfies $dd^c\phi \geq -\omega$ and $-dd^c\phi - \omega \leq R \leq dd^c\phi + \omega$. Since R vanishes outside W_2, we can assume that $W_R \subset W_1 \cap W_2$. Let χ_n be a sequence of smooth functions, with compact support in W_R, with $0 \leq \chi_n \leq 1$ and which increases to the characteristic function of W_R. Define $R_n := \chi_n R$. We still have $-dd^c\phi - \omega \leq R_n \leq dd^c\phi + \omega$.

In order to regularize ϕ, we apply Demailly's method which uses local convolution operators, see [10]. These operators act also on smooth forms R_n and do not change R_n too much. We can find a smooth function ϕ_n on an open subset $W_{1,n}$ of W_1 which vanishes out of an open set $W_{2,n} \supset W_2$ and such that $dd^c\phi_n \geq -\omega$ and $-c(dd^c\phi_n + \omega) \leq R_n \leq c(dd^c\phi_n + \omega)$, where $c > 0$ is a constant independent of ϕ, R and n. The constant c takes into account the loss of positivity in the regularization procedure. We can choose $W_{1,n}$ increasing to W_1 with $W_{1,n} \supset \mathrm{supp}(T)$ and $W_{2,n}$ decreasing to W_2.

Define $T_n := cT \wedge (dd^c\phi_n + \omega)$. For n large enough, this current is supported by $\mathrm{supp}(T) \subset W_{1,n}$ and ϕ_n vanishes outside W. This and Stokes' formula imply that

$$\|T_n\|_W = \langle T_n, \omega^{k-p-1}\rangle_W = c\langle T, \omega^{k-p}\rangle_W = c\|T\|_W.$$

In particular, the mass of T_n is locally bounded uniformly on n. Extracting a subsequence, we can assume that T_n converges to a current T'. Clearly, T' satisfies the lemma. □

Definition 2.7. Let (R_λ) be a family of q-currents on X with $\lambda \in \mathbb{C}$ and $|\lambda| \geq 1$. We say that (R_λ) is *$\star$-negligible* if it can be written as a finite sum of families of q-currents of type

$$\Gamma_\lambda^1 \wedge \ldots \wedge \Gamma_\lambda^q$$

where for each index j, the Γ_λ^j are quasi-continuous forms of the same bidegree $(1, 0)$ or $(0, 1)$ with $\star$-norms bounded uniformly on λ and such that one of the following properties holds

(a) The number of $(1, 0)$-forms is not equal to the number of $(0, 1)$-forms;

(b) For some index j, the $\star$-norm of Γ_λ^j tends to 0 as λ tends to infinity;

(c) For at least $q - 1$ indices j, we can write $\Gamma_\lambda^j = h_\lambda^j S^j$ where S^j is a quasi-continuous form independent of λ with finite $\star$-norm and h_λ^j are quasi-continuous functions, bounded uniformly on λ, whose essential support converges to the empty set as λ tends to infinity.

Here, we say that a family of open sets (U_λ) converges to the empty set if the characteristic function $\mathbf{1}_{U_\lambda}$ converges pointwise to 0. The following lemma justifies the introduction of $\star$-negligible families of currents.

Lemma 2.8. *Let T be a positive closed (p,p)-current on X with support in W_1. Let (R_λ) be a $\star$-negligible family of $(2k-2p)$-forms. Then the mass of $T\wedge R_\lambda$ converges to 0 when λ tends to infinity.*

Proof. We only have to consider the case where R_λ is equal to the wedge-product $\Gamma_\lambda^1\wedge\ldots\wedge\Gamma_\lambda^{2k-2p}$ as in Definition 2.7. If it satisfies property (a) in that definition, then for bidegree reason, we have $T\wedge R_\lambda=0$. So we can, without loss of generality, assume that Γ_λ^j is of bidegree $(1,0)$ when $j\leq k-p$ and of bidegree $(0,1)$ otherwise. Denote for simplicity $\Lambda_\lambda^j:=\overline{\Gamma}_\lambda^{k-p+j}$.

Consider now the case (b). We can assume that $\|\Gamma_\lambda^1\|_\star$ converges to 0. Observe that $\Gamma_\lambda^j\wedge\overline{\Lambda}_\lambda^j$ can be written in a canonical way as a linear combination with constant coefficients of the currents

$$i\Gamma_\lambda^j\wedge\overline{\Gamma}_\lambda^j,\quad i\Lambda_\lambda^j\wedge\overline{\Lambda}_\lambda^j,\quad i(\Gamma_\lambda^j+\Lambda_\lambda^j)\wedge(\overline{\Gamma}_\lambda^j+\overline{\Lambda}_\lambda^j)\quad\text{and}\quad i(\Gamma_\lambda^j+i\Lambda_\lambda^j)\wedge(\overline{\Gamma}_\lambda^j+\overline{i\Lambda}_\lambda^j).$$

Therefore, we can assume that $\Lambda_\lambda^j=\Gamma_\lambda^j$ for $j\geq 2$. Define

$$T_\lambda:=T\wedge(i\Gamma_\lambda^2\wedge\overline{\Gamma}_\lambda^2)\wedge\ldots\wedge(i\Gamma_\lambda^{k-p}\wedge\overline{\Gamma}_\lambda^{k-p}).$$

This is a positive current.

If we apply Lemma 2.6 inductively $k-p-1$ times to $R:=\Gamma_\lambda^j\wedge\overline{\Gamma}_\lambda^j$ we get a positive closed $(k-1,k-1)$-current T'_λ of bounded mass such that $T_\lambda\leq T'_\lambda$. Finally, Cauchy-Schwarz's inequality implies that

$$\|T_\lambda\wedge\Gamma_\lambda^1\wedge\overline{\Lambda}_\lambda^1\|\leq\|T_\lambda\wedge(i\Gamma_\lambda^1\wedge\overline{\Gamma}_\lambda^1)\|^{1/2}\|T_\lambda\wedge(i\Lambda_\lambda^1\wedge\overline{\Lambda}_\lambda^1)\|^{1/2}.$$

Applying again Lemma 2.6 to T'_λ instead of T and to $R=i\Gamma_\lambda^1\wedge\overline{\Gamma}_\lambda^1$ or $R=i\Lambda_\lambda^1\wedge\overline{\Lambda}_\lambda^1$ gives the result.

Assume now that condition (c) is satisfied. We can assume that it holds for $j\neq k-p+1$ and that the functions h_λ^j in this condition are the characteristic functions of open sets W_λ which converge to the empty set. As in the last case, we reduce the problem to the case where $\Gamma_\lambda^j=\Lambda_\lambda^j$ for $2\leq j\leq k-p$; these currents are also equal to S^j restricted to W_λ. Consider the positive current

$$\widetilde{T}:=T\wedge(iS^2\wedge\overline{S}^2)\wedge\ldots\wedge(iS^{k-p}\wedge\overline{S}^{k-p}).$$

We obtain as above

$$\|\widetilde{T}\wedge\Gamma_\lambda^1\wedge\overline{\Lambda}_\lambda^1\|_{W_\lambda}\leq\|\widetilde{T}\wedge(iS^1\wedge\overline{S}^1)\|_{W_\lambda}^{1/2}\|\widetilde{T}\wedge(i\Lambda_\lambda^1\wedge\overline{\Lambda}_\lambda^1)\|^{1/2}.$$

The first factor in the right hand side tends to 0 since this is the mass of a fixed current on open sets which converge to the empty set. The second factor is bounded according to Lemma 2.6. The result follows. □

Lemma 2.9. *Let M be a positive constant. Let φ_1, φ_2 be quasi-psh functions on W_1 which are constant outside W_2 and satisfy $dd^c\varphi_1 \geq -M\omega$ and $dd^c\varphi_2 \geq -M\omega$. Then $\phi := \log(e^{\varphi_1} + e^{\varphi_2})$ is a quasi-psh function on W_1. It is constant outside W_2 and satisfies $dd^c\phi \geq -M\omega$. Moreover, the $\star$-norm of*

$$\Gamma := \frac{e^{\frac{1}{2}(\varphi_1+\varphi_2)}}{e^{\varphi_1} + e^{\varphi_2}}(\partial\varphi_1 - \partial\varphi_2)$$

is bounded by $\sqrt{6\pi M}$.

Proof. Clearly, ϕ is constant outside W_2. Define $\chi(t) := \log(1 + e^t)$. We have $0 \leq \chi'(t) \leq 1$ and $\chi''(t) \geq 0$. Therefore, if $t := \varphi_1 - \varphi_2$ we have

$$\begin{aligned} dd^c\phi &= dd^c(\chi(\varphi_1 - \varphi_2)) + dd^c\varphi_2 \\ &= \chi'(t)(dd^c\varphi_1 - dd^c\varphi_2) + \frac{1}{2\pi}\chi''(t)(i\partial t \wedge \overline{\partial} t) + dd^c\varphi_2 \\ &\geq \chi'(t)dd^c\varphi_1 + (1 - \chi'(t))dd^c\varphi_2. \end{aligned}$$

Recall that $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. Hence, ϕ is quasi-psh and $dd^c\phi \geq -M\omega$. We deduce that the $\star$-norm of $dd^c\phi$ is bounded by $2M$.

A direct computation as above gives that $i\partial\overline{\partial}\phi - i\Gamma \wedge \overline{\Gamma}$ is equal to

$$\frac{e^{\varphi_1}}{e^{\varphi_1} + e^{\varphi_2}}(i\partial\overline{\partial}\varphi_1) + \frac{e^{\varphi_2}}{e^{\varphi_1} + e^{\varphi_2}}(i\partial\overline{\partial}\varphi_2).$$

The $\star$-norm of the last sum is bounded by $4\pi M$ because $i\partial\overline{\partial}\varphi_1$ and $i\partial\overline{\partial}\varphi_2$ satisfy the same property. This gives the last estimate in the lemma. □

We will use test forms with finite $\star$-norms. The results obtained above permit to bound some integrals without knowing the L^∞-norm of test forms which is not controlled by the $\star$-norm. Therefore, specific test forms with bounded $\star$-norms can be used to study singularities of currents. We now describe a situation that we will consider in the next section, in particular, when dealing with a blow-up.

Let V and V' be submanifolds of X of dimension l and l' respectively such that $V' \subset V \subset W_2$. So $V \cap W_1$ is relatively compact in X. In $W_1 \cap W_2$ consider a chart which is identified with the polydisc $2\mathbb{D}^k$ on which V and V' are equal respectively to $2\mathbb{D}^l \times \{0\}$ and $2\mathbb{D}^{l'} \times \{0\}$. We will use in this polydisc the standard coordinates $x = (x^1, x^2, x^3)$ with $x^1 = (x_1, \ldots, x_{l'})$, $x^2 = (x_{l'+1}, \ldots, x_l)$ and $x^3 = (x_{l+1}, \ldots, x_k)$. The following lemmas introduce useful families of test forms.

Lemma 2.10. *Let $\Gamma_{m,j}$ denote the $(1,0)$-form supported by $\mathbb{D}^k$ and defined by*

$$\Gamma_{m,j} := \frac{x_j dx_m - x_m dx_j}{\|x^3\|^2} \quad \textit{for} \quad l + 1 \leq m, j \leq k.$$

Then $\Gamma_{m,j}$ has finite $\star$-norm.

We first introduce some notations. Let $\sigma : \widehat{X} \to X$ denote the blow-up of X along V and define $\widehat{V} := \sigma^{-1}(V)$. By Blanchard's theorem [4], if U is an open subset of $\widehat{X}$ such that $U \cap \widehat{V}$ is relatively compact in $\widehat{X}$, then U is a Kähler manifold. Let $\widehat{\omega}$ be a Kähler form on a neighbourhood of $\widehat{W_1} := \sigma^{-1}(W_1)$. We can choose $\widehat{\omega}$ so that $\sigma_*(\widehat{\omega})$ is equal to a constant times ω outside W_2. The current $\sigma_*(\widehat{\omega})$ is positive closed and has positive Lelong number along V (if V is a hypersurface then $\sigma = \mathrm{id}$; we replace $\sigma_*(\widehat{\omega})$ by $\omega + [V]$). Multiplying $\widehat{\omega}$ with a constant allows to assume that the Lelong number of $\sigma_*(\widehat{\omega})$ along V is equal to 1 or equivalently, if $\widehat{V} := \sigma^{-1}(V)$ is the exceptional hypersurface then $\sigma^*(\sigma_*(\widehat{\omega})) = \widehat{\omega} + [\widehat{V}]$.

Since $\sigma_*(\widehat{\omega})$ is smooth outside V, we can find a negative quasi-psh function φ on W_1 which vanishes outside W_2 and such that $dd^c\varphi - \sigma_*(\widehat{\omega})$ is a smooth form. Fix a constant $c_0 > 1$ large enough such that $dd^c\varphi \geq \sigma_*(\widehat{\omega}) - (c_0-1)\omega$ and define $\alpha := dd^c\varphi + c_0\omega$. This form is larger than $\sigma_*(\widehat{\omega}) + \omega$ and its restriction to W_1 has finite $\star$-norm.

We cover $\sigma^{-1}(\mathbb{D}^k)$ with $k-l$ charts. We describe only one of them. The other ones are obtained by permuting the coordinates. The chart we consider is denoted by $\widehat{D}$ and is given with local coordinates $z = (z_1, \ldots, z_k)$ with $|z_i| < 2$ and such that

$$\sigma(z) = (z_1, \ldots, z_l, z_{l+1}z_k, \ldots, z_{k-1}z_k, z_k).$$

On this chart, $\widehat{V}$ is equal to $\{z_k = 0\}$. Since $dd^c(\varphi \circ \sigma) - [\widehat{V}]$ is a smooth form and $dd^c \log|z_k| = [\widehat{V}]$, the function $\varphi \circ \sigma - \log|z_k|$ is smooth. We deduce that $\varphi - \log\|x^3\|$ is a bounded function.

Proof of Lemma 2.10. When V is a hypersurface, i.e. $l = k-1$, we have $\Gamma_{m,j} = 0$. Consider the higher codimension case. Observe that $2i\partial\overline{\partial}\log\|x^3\|$ is equal to the sum of $i\Gamma_{m,j} \wedge \Gamma_{m,j}$. So we only have to check that the form $i\partial\overline{\partial}\log\|x^3\|$ restricted to $\mathbb{D}^k$ has finite $\star$-norm.

Using the local coordinates z introduced above, we see that $\sigma^*(i\partial\overline{\partial}\log\|x^3\|)$ is bounded by $2\pi[\widehat{V}]$ plus a smooth $(1,1)$-form. It follows that $i\partial\overline{\partial}\log\|x^3\|$ is bounded by a constant times α. Therefore, $i\partial\overline{\partial}\log\|x^3\|$ has finite $\star$-norm. $\square$

Denote by A_λ the map $(x^1, x^2, x^3) \mapsto (x^1, x^2, \lambda x^3)$ for $\lambda \in \mathbb{C}^*$. We will be concerned with $|\lambda| \to \infty$. Therefore, in what follows, we assume that $|\lambda| \geq 1$.

Lemma 2.11. *Let R (resp. Γ) be a quasi-continuous form essentially supported in $\mathbb{D}^k$ and of bidegree $(1,1)$ (resp. $(1,0)$ or $(0,1)$). Assume that their coefficients have modulus smaller or equal to 1. Then the forms $(A_\lambda)^*(R)$ and $(A_\lambda)^*(\Gamma)$ on $\mathbb{D}^l \times \lambda^{-1}\mathbb{D}^{k-l}$ have $\star$-norms bounded by a constant independent of R, Γ and λ.*

Proof. Observe that the estimate on $\|(A_\lambda)^*(\Gamma)\|_\star$ can be deduced from the estimate on $\|(A_\lambda)^*(R)\|_\star$ applied to $R := \Gamma \wedge \overline{\Gamma}$. So it is enough to bound the $\star$-norm of $(A_\lambda)^*(R)$. If R does not contain terms with dx_j or $d\overline{x}_j$ with $j \geq l+1$, then $(A_\lambda)^*(R)$ has bounded coefficients and its $\star$-norm is clearly bounded.

Moreover, since we can bound the real and imaginary parts of $dx_m \wedge d\overline{x}_j$ by $idx_m \wedge d\overline{x}_m + idx_j \wedge d\overline{x}_j$, we only have to consider the case where $R = i\partial\overline{\partial}\|x^3\|^2$.

We construct now a function ϕ satisfying estimates as in Definition 2.1. Define $s := \log|\lambda|$. Recall that $\varphi - \log\|x^3\|$ is a bounded function. Fix a constant $A \geq 1$ large enough such that $-A \leq \varphi - \log\|x^3\| \leq A$, where the function φ is defined above. We only have to consider the case where s is large enough, e.g. $s \geq 3A$. Observe that since A is large enough, we have $\varphi \leq -s + 2A$ on $\mathbb{D}^l \times \lambda^{-1}\mathbb{D}^{k-l}$.

Let χ be a convex increasing function on $\mathbb{R}$ such that $\chi(t) = t$ for $t \geq -s+3A$, $0 \leq \chi' \leq 1$ everywhere and $\chi''(t) = e^{2t+2s-5A}$ for $t \leq -s+2A$. Define $\phi := c^{-1}\chi\circ\varphi$ for a fixed constant $c \gg c_0$ large enough. It is clear that ϕ vanishes outside W_2. A direct computation gives

$$i\partial\overline{\partial}\phi = c^{-1}\chi''(\varphi)i\partial\varphi \wedge \overline{\partial}\varphi + c^{-1}\chi'(\varphi)i\partial\overline{\partial}\varphi.$$

The first term in the last sum is positive. The second one is bounded below by $-2\pi c^{-1}c_0\omega$. Therefore, $i\partial\overline{\partial}\phi \geq -\omega$ and ϕ is quasi-psh.

We prove now that $e^{2s}i\partial\overline{\partial}\|x^3\|^2 \leq c^3(i\partial\overline{\partial}\phi + \alpha)$ on the open subset of $\mathbb{D}^k$ where $\varphi < -s + 2A$. This property implies that the form $(A_\lambda)^*(i\partial\overline{\partial}\|x^3\|^2)$ on $\mathbb{D}^k \times \lambda^{-1}\mathbb{D}^{k-l}$ has bounded $\star$-norm because of the choice of s and of A. The idea is to pull-back the forms by σ and check the inequality in the chart $\widehat{D}$ that we have described.

Define $\widehat{\varphi} := \varphi \circ \sigma$ and $\widehat{\phi} := \phi \circ \sigma = c^{-1}\chi \circ \widehat{\varphi}$. Since $\widehat{\varphi} - \log|z_k|$ is a smooth function, the form $\gamma := \partial(\widehat{\varphi} - \log|z_k|)$ is smooth. Recall also that $i\partial\overline{\partial}\widehat{\varphi} \geq -2\pi c_0\sigma^*(\omega)$ and $\widehat{\varphi} \geq \log|z_k| - A$. Therefore, when $\widehat{\varphi}(z) < -s + 2A$, a direct computation as above gives

$$\begin{aligned} c^3(i\partial\overline{\partial}\widehat{\phi} + \sigma^*(\alpha)) &\geq c^2\chi''(\widehat{\varphi})i\partial\widehat{\varphi} \wedge \overline{\partial}\widehat{\varphi} + c^2\chi'(\widehat{\varphi})i\partial\overline{\partial}\widehat{\varphi} + c^3\sigma^*(\alpha) \\ &\geq c^2e^{2s-7A}|z_k|^2 i\partial\widehat{\varphi} \wedge \overline{\partial}\widehat{\varphi} - 2c^2\pi c_0\sigma^*(\omega) + c^3(\widehat{\omega} + \sigma^*(\omega)) \\ &\geq c^2e^{2s-7A}|z_k|^2 i(z_k^{-1}dz_k + \gamma) \wedge (\overline{z}_k^{-1}d\overline{z}_k + \overline{\gamma}) + c^3\widehat{\omega}. \end{aligned}$$

We also have

$$\begin{aligned} 2i(z_k^{-1}dz_k + \gamma) \wedge (\overline{z}_k^{-1}d\overline{z}_k + \overline{\gamma}) &= i(z_k^{-1}dz_k + 2\gamma) \wedge (\overline{z}_k^{-1}d\overline{z}_k + 2\overline{\gamma}) \\ &\quad + |z_k|^{-2}idz_k \wedge d\overline{z}_k - 2i\gamma \wedge \overline{\gamma} \\ &\geq |z_k|^{-2}idz_k \wedge d\overline{z}_k - 2i\gamma \wedge \overline{\gamma}. \end{aligned}$$

Since $e^{2s-7A}|z_k|^2\gamma \wedge \overline{\gamma}$ is a bounded form on the considered domain and because the constant c is large enough, we deduce from the inequalities above that

$$c^3(i\partial\overline{\partial}\widehat{\phi} + \sigma^*(\alpha)) \geq ce^{2s}idz_k \wedge d\overline{z}_k + c\widehat{\omega}.$$

On the other hand, one can find bounded forms θ_i on $\mathbb{D}^l \times \lambda^{-1}\mathbb{D}^{k-l}$ such that

$$\sigma^*(e^{2s}dd^c\|x^3\|^2) = e^{2s}dd^c|z_k|^2 + e^s dz_k \wedge \theta_1 + e^s d\overline{z}_k \wedge \theta_2 + \theta_3.$$

Cauchy-Schwarz's inequality implies that the last sum is bounded above by $2e^{2s}dd^c|z_k|^2 + \theta_4$ for some bounded form θ_4. We conclude that

$$\sigma^*(e^{2s}dd^c\|x^3\|^2) \leq c^3(i\partial\overline{\partial}\widehat{\phi} + \sigma^*(\alpha))$$

on $\widehat{D} \cap \sigma^{-1}(\mathbb{D}^l \times \lambda^{-1}\mathbb{D}^{k-l})$. Hence,

$$e^{2s}dd^c\|x^3\|^2 \leq c^3(i\partial\overline{\partial}\phi + \alpha)$$

on $\mathbb{D}^l \times \lambda^{-1}\mathbb{D}^{k-l}$. This completes the proof of the lemma. $\square$

Lemma 2.12. *Let $\Gamma'_{m,j}$ be the $(1,0)$-form on $\mathbb{C}^k$ given by*

$$\Gamma'_{m,j} := \frac{x_j dx_m - x_m dx_j}{\|x^2\|^2 + \|x^3\|^2} \quad \textit{for} \quad l'+1 \leq m, j \leq k.$$

Then the restriction of $(A_\lambda)^*(\Gamma'_{m,j})$ to $\mathbb{D}^k$ has $\star$-norm bounded by a constant which does not depend on λ, m and j with $|\lambda| \geq 1$.

Proof. We have to bound the $\star$-norm of $(A_\lambda)^*(i\Gamma'_{m,j}\wedge\overline{\Gamma}'_{m,j})$. Observe that the form $i\Gamma'_{m,j}\wedge\overline{\Gamma}'_{m,j}$ is positive and bounded by the form $R := i\partial\overline{\partial}\log(\|x^2\|^2+\|x^3\|^2)$. So it is enough to bound the $\star$-norm of the restriction of $(A_\lambda)^*(R) = i\partial\overline{\partial}\log(\|x^2\|^2 + |\lambda|^2\|x^3\|^2)$ to $\mathbb{D}^k$. Write $\widetilde{\varphi}_1 := \log[(|\lambda|^2-1)\|x^3\|^2]$ and $\widetilde{\varphi}_2 := \log(\|x^2\|^2+\|x^3\|^2)$. We have $(A_\lambda)^*(R) = i\partial\overline{\partial}\log(e^{\widetilde{\varphi}_1} + e^{\widetilde{\varphi}_2})$.

With notations as in Lemma 2.10, we can write $i\partial\overline{\partial}\widetilde{\varphi}_1$ as a finite combination of $\Gamma_{m,j}\wedge\overline{\Gamma}_{m,j}$. Therefore, $i\partial\overline{\partial}\widetilde{\varphi}_1$ has bounded $\star$-norm. The same arguments applied to V' instead of V imply that the $\star$-norm of $i\partial\overline{\partial}\widetilde{\varphi}_2$ is bounded. Recall that the function φ was defined above using the blow-up $\sigma : \widehat{X} \to X$ of X along V. Let φ' be the function obtained in the same way by replacing V with V'. Define also $\varphi_1 := 2\varphi + \log(|\lambda|^2-1)$ and $\varphi_2 := 2\varphi'$. Observe that $i\partial\overline{\partial}\varphi_1$ and $i\partial\overline{\partial}\varphi_2$ restricted to $\mathbb{D}^k$ have $\star$-norms bounded independently of λ.

Using the coordinates z on $\widehat{D}$ introduced above, we see that $(\varphi_1-\widetilde{\varphi}_1)\circ\sigma$ is the potential of a smooth form. So, it is a smooth function. We deduce that $\varphi_1-\widetilde{\varphi}_1$ is a bounded function and $\partial\varphi_1 - \partial\widetilde{\varphi}_1$ is the push-forward by σ of a combination of dz_j with bounded coefficients. Therefore, $\partial\varphi_1-\partial\widetilde{\varphi}_1$ is equal to the sum of a bounded form and a combination with bounded coefficients of $\Gamma_{m,j}$. We deduce that this form has bounded $\star$-norm. In the same way, we obtain that $\varphi_2-\widetilde{\varphi}_2$ is a bounded function and $\partial\varphi_2-\partial\widetilde{\varphi}_2$ has bounded $\star$-norm. These functions and forms do not depend on λ.

Define

$$\Gamma := \frac{e^{\frac{1}{2}(\varphi_1+\varphi_2)}}{e^{\varphi_1}+e^{\varphi_2}}(\partial\varphi_1-\partial\varphi_2) \quad \text{and} \quad \widetilde{\Gamma} := \frac{e^{\frac{1}{2}(\widetilde{\varphi}_1+\widetilde{\varphi}_2)}}{e^{\widetilde{\varphi}_1}+e^{\widetilde{\varphi}_2}}(\partial\widetilde{\varphi}_1-\partial\widetilde{\varphi}_2).$$

The coefficients involving the exponential in the last line are smaller than 1 since the exponential function is convex. We deduce from the above properties of $\partial\varphi_j - \partial\widetilde{\varphi}_j$ that $\widetilde{\Gamma}$ is equal to a $\star$-bounded form plus the form

$$\widetilde{\Gamma}' := \frac{e^{\frac{1}{2}(\widetilde{\varphi}_1+\widetilde{\varphi}_2)}}{e^{\widetilde{\varphi}_1}+e^{\widetilde{\varphi}_2}}(\partial\varphi_1 - \partial\varphi_2).$$

Moreover, $\widetilde{\Gamma}'$ is equal to a bounded function times Γ which is $\star$-bounded according to Lemma 2.9. It follows that $\|\widetilde{\Gamma}\|_\star$ is also bounded.

A computation as in Lemma 2.9 shows that

$$(A_\lambda)^*(R) = i\widetilde{\Gamma}\wedge\overline{\widetilde{\Gamma}} + \frac{e^{\widetilde{\varphi}_1}}{e^{\widetilde{\varphi}_1}+e^{\widetilde{\varphi}_2}}(i\partial\overline{\partial}\widetilde{\varphi}_1) + \frac{e^{\widetilde{\varphi}_2}}{e^{\widetilde{\varphi}_1}+e^{\widetilde{\varphi}_2}}(i\partial\overline{\partial}\widetilde{\varphi}_2).$$

By Lemma 2.10, $i\partial\overline{\partial}\widetilde{\varphi}_1$ and $i\partial\overline{\partial}\widetilde{\varphi}_2$ have bounded $\star$-norms. It follows that $(A_\lambda)^*(R)$ has also a bounded $\star$-norm. This completes the proof of the lemma. □

Lemma 2.13. *Let R be a smooth q-form with compact support in $\mathbb{D}^k$. Let $t \in \mathbb{C}$ be a fixed constant such that $|t| \geq 1$. Assume that V is a hypersurface, i.e. $l = k-1$ and $\sigma = \mathrm{id}$. Then there are smooth $(q-1)$-forms Θ_λ with compact support in $\mathbb{D}^k$ such that the family of forms*

$$(A_{t\lambda})^*(R) - (A_\lambda)^*(R) - d\Theta_\lambda$$

on $\mathbb{D}^l \times \lambda^{-1}\mathbb{D}^{k-l}$ is $\star$-negligible.

Proof. The map A_λ is given by $(x_1, \ldots, x_{k-1}, x_k) \mapsto (x_1, \ldots, x_{k-1}, \lambda x_k)$. Solving the d-equation on the complex lines where $x_1, \ldots, x_{k-1}$ are constant, we obtain a smooth $(q-1)$-form Θ_λ with compact support in $\mathbb{D}^k$ such that $A^*_{t\lambda}(R) - A^*_\lambda(R) - d\Theta_\lambda$ does not contain terms with $dx_k \wedge d\overline{x}_k$.

Finally, observe that dx_j and $d\overline{x}_j$ are invariant under the actions of $A^*_{t\lambda}$ for $j \leq k-1$. By Lemma 2.10, they have finite $\star$-norms. Therefore, the family of forms in the lemma satisfies the property (c) in Definition 2.7. □

Let $x' := (x^1, x^2)$ and $x'' := x^3$. For $m \geq 0$, denote by $O^*(\|x''\|^m)$ a function (resp. a 1-form) which is continuous outside V and is equal to (resp. whose coefficients are equal to) $O(\|x''\|^m)$ when $x'' \to 0$. Recall that a function is equal to $O(\|x''\|^m)$ if its modulus is bounded by a constant times $\|x''\|^m$. We will use functions and forms depending on a parameter λ and we always assume that the constant is independent of λ. Denote also by $O^{**}(\|x''\|^m)$ the sum of a 1-form with $O^*(\|x''\|^m)$ coefficients and a combination of $dx'', d\overline{x}''$ with $O^*(\|x''\|^{m-1})$ coefficients for $m \geq 1$. A vector or a matrix whose coefficients satisfy the same property is denoted with the same notation.

Consider now a bi-Lipschitz map τ from a neighbourhood U of $\overline{\mathbb{D}}^k \cap V$ to another neighbourhood of $\overline{\mathbb{D}}^k \cap V$. We assume that τ is smooth outside $U \cap V$ and that its restriction to $U \cap V$ is identity. In the following expressions, we consider x' and x'' as line matrices.

Definition 2.14. We say that τ is *admissible* if there is an $O^*(1)$ matrix $a(x)$ on U such that

$$\tau = \big(x' + x''a(x), x''\big) + O^*(\|x''\|^2)$$

and

$$d\tau(x) = \big(dx' + dx''a(x) + O^*(\|x''\|), dx'' + O^{**}(\|x''\|^2)\big)$$

when $x'' \to 0$.

Equivalently, there are $O^*(1)$ matrices $a_1(x)$ and $a_2(x)$ on U such that

$$\tau = \big(x^1 + x^3 a_1(x), x^2 + x^3 a_2(x), x^3\big) + O^*(\|x^3\|^2)$$

and

$$\begin{aligned} d\tau(x) \ = \ & \big(dx^1 + dx^3 a_1(x) + O^*(\|x^3\|), dx^2 + dx^3 a_2(x) + O^*(\|x^3\|), \\ & \quad dx^3 + O^{**}(\|x^3\|^2)\big) \end{aligned}$$

when $x^3 \to 0$.

Note that the differential of a smooth and admissible map is $\mathbb{C}$-linear at every point of $U \cap V$ but in general it does not depend holomorphically on the point. The map $\tau = (x' + x''a(x), x'')$ with a holomorphic is admissible, but we will need later global admissible maps which are not necessarily holomorphic, even not smooth.

Definition 2.15. Let (R_λ) be a family of (q,q)-currents on X with $\lambda \in \mathbb{C}$ and $|\lambda| \geq 1$. We say that this family is $\star$-*principal* if it can be written as a finite sum of families of (q,q)-currents of type

$$\Gamma^1_\lambda \wedge \ldots \wedge \Gamma^{2q}_\lambda,$$

where the Γ^j_λ are quasi-continuous forms with $\star$-norms bounded uniformly on λ such that q of them are of bidegree $(1,0)$ and q are of bidegree $(0,1)$.

Let τ be an admissible map as above. Denote by $\sigma' : \widehat{X}' \to X$ the blow-up along V'. We will need some test forms which are the push-forward of smooth forms by σ'. We have the following lemma.

Lemma 2.16. *Let R be a smooth $2q$-form with compact support in $\sigma'^{-1}(\mathbb{D}^k)$. Let R' be the bidegree (q,q) component of R and define $R_\lambda := (A_\lambda)^* \sigma'_*(R')$. Then the family R_λ is $\star$-principal and the family $\tau^*(A_\lambda)^*\sigma'_*(R) - R_\lambda$ is $\star$-negligible.*

Proof. We cover $\sigma'^{-1}(\mathbb{D}^k)$ with a finite number of charts. Using a partition of unity, we can assume that R is supported by one of these charts. We will only work in the chart $\widehat{D}'$ that we describe now. The result holds for the other ones because they are obtained from $\widehat{D}'$ just by using some permutations of indices.

We have on $\widehat{D}'$ holomorphic coordinates $w = (w_1, \ldots, w_k)$ with $|w_i| < 2$ such that

$$\sigma'(w) = (w_1, \ldots, w_{l'}, w_{l'+1}w_k, \ldots, w_{k-1}w_k, w_k).$$

Denote by x the image of w by σ'. We have $w_j = x_j/x_k$ for $l' + 1 \leq j \leq k - 1$ and $w_j = x_j$ otherwise. We also have $\|x^2\| \lesssim |x_k|$ and $\|x^3\| \lesssim |x_k|$ on $\sigma'(\widehat{D}')$. This implies that $\|x^2\| \lesssim |\lambda||x_k|$ and $\|x^3\| \lesssim |x_k| \lesssim |\lambda|^{-1}$ on $(A_\lambda)^{-1}\sigma'(\widehat{D}')$ and on $\tau^{-1}(A_\lambda)^{-1}\sigma'(\widehat{D}')$ because τ is bi-Lipschitz. The later sets contain respectively the support of $(A_\lambda)^*\sigma'_*(R)$ and the support of $\tau^*(A_\lambda)^*\sigma'_*(R)$.

In order to obtain the result, we first study the actions of $(A_\lambda)^*\sigma'_*$ and of $\tau^*(A_\lambda)^*\sigma'_*$ on smooth functions and on linear 1-forms. The form R is built using these functions and 1-forms. Let g be a smooth function with compact support in $\widehat{D}'$. If we define

$$w_{x,\lambda} := \sigma'^{-1}(A_\lambda(x)) = (x^1, \lambda^{-1}x_k^{-1}x^2, x_k^{-1}x_{l+1}, \ldots, x_k^{-1}x_{k-1}, x_k),$$

then

$$\tau^*(A_\lambda)^*\sigma'_*(g)(x) - (A_\lambda)^*\sigma'_*(g)(x) = g(w_{\tau(x),\lambda}) - g(w_{x,\lambda}).$$

Since τ is admissible, the above estimates on $\|x^2\|$ and $\|x^3\|$ imply that $\|w_{\tau(x),\lambda} - w_{x,\lambda}\| = O(\lambda^{-1})$. The smoothness of g implies that the functions in the previous identity are uniformly bounded by a constant times $|\lambda|^{-1}$.

Consider now the forms $\tau^*(A_\lambda)^*\sigma'_*(dw_j)$ and $\tau^*(A_\lambda)^*\sigma'_*(d\overline{w}_j)$. We will discuss the case of dw_j; the other case is treated similarly. Observe that since τ is admissible, for λ large enough, the supports of the considered forms are contained in $\mathbb{D}^l \times 2\lambda^{-1}\mathbb{D}^{k-l}$. By Lemma 2.11 applied to $\lambda/2$ instead of λ, on the considered domains, bounded forms have bounded $\star$-norms and the $\star$-norm of an $O^{**}(\|x^3\|)$ 1-form is of order $O(\lambda^{-1})$ as λ tends to infinity. We will use these properties and the admissibility of τ several times in the discussion below.

Define $w^1 := (w_1, \ldots, w_{l'})$. We have

$$\tau^*(A_\lambda)^*\sigma'_*(dw^1) = \tau^*(dx^1) = dx^1 + O^{**}(\|x^3\|).$$

Since the components of $(A_\lambda)^*\sigma'_*(dw^1) = dx^1$ are bounded forms, they have bounded $\star$-norms. The $\star$-norm of the components of $O^{**}(\|x^3\|)$ on the considered domains tends to 0 as λ tends to infinity. So the $\star$-norm of $\tau^*(A_\lambda)^*\sigma'_*(dw^1) - (A_\lambda)^*\sigma'_*(dw^1)$ tends to 0.

For $j = k$, we have

$$\tau^*(A_\lambda)^*\sigma'_*(dw_k) = \lambda dx_k + \lambda O^{**}(\|x^3\|^2) = (A_\lambda)^*\sigma'_*(dw_k) + O^{**}(\|x^3\|).$$

As we already wrote above, the form $(A_\lambda)^*\sigma'_*(dw_k) = \lambda dx_k$ has bounded $\star$-norm and the $\star$-norm of $O^{**}(\|x^3\|)$ tends to 0. So the $\star$-norm of $\tau^*(A_\lambda)^*\sigma'_*(dw_k) - (A_\lambda)^*\sigma'_*(dw_k)$ tends to 0.

Assume that $l+1 \leq j \leq k-1$. We have

$$(A_\lambda)^* \sigma'_*(dw_j) = (A_\lambda)^* \Big(\frac{x_k dx_j - x_j dx_k}{x_k^2} \Big) = \frac{x_k dx_j - x_j dx_k}{x_k^2} \cdot$$

By Lemma 2.10, this form has bounded $\star$-norm. As above, the admissibility of τ implies that the $\star$-norm of $\tau^*(A_\lambda)^*\sigma'_*(dw_j) - (A_\lambda)^*\sigma'_*(dw_j)$ tends to 0.

Consider now the remaining case where $l'+1 \leq j \leq l$. We have

$$(A_\lambda)^* \sigma'_*(dw_j) = (A_\lambda)^* \Big(\frac{x_k dx_j - x_j dx_k}{x_k^2} \Big) = \frac{x_k dx_j - x_j dx_k}{\lambda x_k^2} \cdot$$

Since $\|x^2\| \lesssim |\lambda||x_k|$, by Lemma 2.12, this form has bounded $\star$-norm. Using the description of $d\tau$ and that $\|x^2\| \lesssim \lambda |x_k|$, we obtain that

$$\tau^*(A_\lambda)^*\sigma'_*(dw_j) - \frac{x_k dx_j - x_j dx_k}{\lambda x_k^2}$$

is equal to $O^{**}(\|x^3\|)$ plus a linear combination of the forms considered in the previous case with $O(\lambda^{-1})$ coefficients. Therefore, the $\star$-norm of $\tau^*(A_\lambda)^*\sigma'_*(dw_j) - (A_\lambda)^*\sigma'_*(dw_j)$ tends to 0.

We can now apply the above discussion to each component of R written in w-coordinates. It is easy to deduce that the family R_λ is $\star$-principal and $\tau^*(A_\lambda)^*\sigma'_*(R) - R_\lambda$ is $\star$-negligible and is a sum of forms satisfying (a) or (b) in Definition 2.7. □

Lemma 2.17. *Let R be a continuous $2q$-form with compact support in $\sigma'^{-1}(\mathbb{D}^k)$ such that $\|R\|_\infty \leq 1$. Then the $\star$-norm of $\tau^*(A_\lambda)^*\sigma'_*(R)$ is bounded by a constant independent of λ and of R.*

Proof. Observe that the computation in the last lemma is valid in this case except for the estimate on $\tau^*(A_\lambda)^*\sigma'_*(g)$ when g is only continuous and bounded by 1. However, we only need here that $\tau^*(A_\lambda)^*\sigma'_*(g)$ is bounded by 1. For λ large enough, R'_λ is supported by $\mathbb{D}^l \times 2\lambda^{-1}\mathbb{D}^{k-l}$. Therefore, we easily deduce from the computation in the last lemma that the $\star$-norm of $\tau^*(A_\lambda)^*\sigma'_*(R)$ is bounded by a constant independent of λ and of R. □

Let τ be a bi-Lipschitz map from a neighbourhood U of $\overline{\mathbb{D}}^k \cap V$ to another neighbourhood of $\overline{\mathbb{D}}^k \cap V$. We assume that τ is smooth outside $U \cap V$ and that its restriction to $U \cap V$ is identity.

Definition 2.18. We say that τ is *almost-admissible* if

$$\tau = \big(x' + O^*(\|x''\|), x'' + O^*(\|x''\|^2)\big)$$

and

$$d\tau(x) = \big(dx' + O^{**}(\|x''\|), dx'' + O^{**}(\|x''\|^2)\big)$$

when $x'' \to 0$.

Remark 2.19. Let τ be an almost-admissible map as above. When τ is smooth, its differential at a point of V is not necessarily $\mathbb{C}$-linear. Let R be a smooth $2q$-form with compact support in $\mathbb{D}^k$ and let R' be its component of bidegree (q,q). Define $R_\lambda := (A_\lambda)^*(R')$. As in Lemmas 2.16, we obtain that the family $\tau^*(A_\lambda)^*(R) - R_\lambda$ is $\star$-negligible. If the coefficients of R are bounded by 1, as in Lemmas 2.17, the $\star$-norm of $\tau^*(A_\lambda)^*(R)$ is bounded by a constant independent of λ and of R.

We close this section with a technical lemma that we will use in the next sections. Let W and $\widetilde{W}$ be Kähler manifolds of dimension k. Let V and $\widetilde{V}$ be complex submanifolds of dimension l of W and $\widetilde{W}$ respectively. Consider a bi-Lipschitz map $\tau : W \to \widetilde{W}$ which is smooth outside V, preserves the orientation and such that $\tau(V) = \widetilde{V}$. Denote by Γ the graph of τ in $W\times\widetilde{W}$, $\Pi : W\times\widetilde{W} \to W$ and $\widetilde{\Pi} : W \times \widetilde{W} \to \widetilde{W}$ the canonical projections.

The integration on Γ defines a closed current $[\Gamma]$ of order 0. This can be seen using de Rham regularization theorem for currents. If θ is a smooth q-form on W then $\tau_*(\theta)$ is a bounded form which is equal in the sense of currents to $\widetilde{\Pi}_*([\Gamma]\wedge\Pi^*(\theta))$. In particular, if θ is closed or exact, so is $\tau_*(\theta)$. It follows that τ defines a morphism τ_* from the cohomology ring $\oplus H^*_V(W,\mathbb{C})$ (resp. $\oplus H^*_c(W,\mathbb{C})$ and $\oplus H^*(W,\mathbb{C})$) to the cohomology ring $\oplus H^*_{\widetilde{V}}(\widetilde{W},\mathbb{C})$ (resp. $\oplus H^*_c(\widetilde{W},\mathbb{C})$ and $\oplus H^*(\widetilde{W},\mathbb{C})$). The same property holds for τ^{-1} and gives a morphism τ^*.

Let Z be a smooth oriented manifold of dimension q and let $\rho : Z \to W$ be a Lipschitz proper map. We can define a current Δ_ρ of order 0 on W by $\langle\Delta_\rho,\theta\rangle := \langle Z,\rho^*(\theta)\rangle$ for smooth q-forms θ with compact support in W. We can see using the graph of ρ that this current is closed. De Rham regularization theorem implies that $\tau_*\{\Delta_\rho\} = \{\Delta_{\tau\circ\rho}\}$. Since τ is bi-Lipschitz, we also obtain that $\{\Delta_\rho\} = \tau^*\{\Delta_{\tau\circ\rho}\}$. Hence, $\tau_*\circ\tau^* = \mathrm{id}$. Note that we have $\tau_*\{V\} = \{\widetilde{V}\}$ and $\tau^*\{\widetilde{V}\} = \{V\}$. As above, the map $\tau_{|V} : V \to \widetilde{V}$ induces also isomorphisms $(\tau_{|V})_*$ and $(\tau_{|V})^*$ between the cohomology rings on V and on $\widetilde{V}$.

Lemma 2.20. *Let T be a positive closed (p,p)-current on W without mass on V such that* $\mathrm{supp}(T)\cap V$ *is compact. Then the current $(\tau_{|W\setminus V})_*(T)$ has finite mass on any compact subset of $\widetilde{W}$. Let $\tau_*(T)$ denote the extension of $(\tau_{|W\setminus V})_*(T)$ by 0 to a current on $\widetilde{W}$. Then $\tau_*(T)$ is a closed current such that $\{\tau_*(T)\} = \tau_*\{T\}$ and $\{\tau_*(T)\}_{|\widetilde{V}} = (\tau_{|V})_*(\{T\}_{|V})$.*

Proof. Let θ be a smooth $(2k-2p)$-form with compact support on $\widetilde{W}$. We have

$$\langle(\tau_{|W\setminus V})_*(T),\theta\rangle = \langle T,\tau^*(\theta)\rangle_{W\setminus V}.$$

If the coefficients of θ are bounded, $\tau^*(\theta)$ satisfies the same property because τ is Lipschitz. It follows that the above integrals are bounded and then $(\tau_{|W\setminus V})_*(T)$

has bounded mass on compact subsets of $\widetilde{W}$. We can extend it by 0 to a current $\tau_*(T)$ on $\widetilde{W}$.

We show that this current is closed. The problem concerns only the points near V. Assume that $\theta = d\gamma$ with γ smooth supported in a compact set K in $\widetilde{W}$. We have to prove that the above integrals vanish. Using a partition of unity, we can assume that $\tau^{-1}(K)$ is contained in the chart $\mathbb{D}^k$ as above. Let χ be a function on $\mathbb{D}^l \times \mathbb{C}^{k-l}$ which vanishes in a neighbourhood of $V = \mathbb{D}^l \times \{0\}$ and is equal to 1 outside $\mathbb{D}^k$. Since T is closed, the considered integrals are equal to

$$\lim_{\lambda\to\infty} \big\langle T, (\chi\circ A_\lambda) d(\tau^*(\gamma))\big\rangle = \lim_{\lambda\to 0} -\big\langle T, (A_\lambda)^*(d\chi)\wedge\tau^*(\gamma)\big\rangle.$$

We show that $\langle T, (A_\lambda)^*(\partial\chi)\wedge\tau^*(\gamma)\rangle$ tends to 0 as $\lambda\to\infty$. This and an analogous property with $\overline{\partial}\chi$ instead of $\partial\chi$ give the result. We only have to consider the bidegree $(k-p-1, k-p)$ part of $\tau^*(\gamma)$ because T is of bidegree (p,p). Since this is a bounded form, we can write it as a finite combination of forms of type $\beta\wedge\Theta$, where β is a $(0,1)$-form and Θ a positive $(k-p-1, k-p-1)$-form, both are bounded and smooth outside V. Without loss of generality, we can replace $\tau^*(\gamma)$ by $\beta\wedge\Theta$.

Define $\Gamma_\lambda := (A_\lambda)^*(\partial\chi)$. We obtain from the Cauchy-Schwarz's inequality

$$|\langle T, \Gamma_\lambda\wedge\beta\wedge\Theta\rangle|^2 \leq \langle T, i\overline{\beta}\wedge\beta\wedge\Theta\rangle_{\mathbb{D}^l\times\lambda^{-1}\mathbb{D}^{k-l}}\langle T, i\Gamma_\lambda\wedge\overline{\Gamma}_\lambda\wedge\Theta\rangle.$$

The first factor in the right hand side of the last inequality tends to 0 since T has no mass on V. The second one is bounded according to Lemmas 2.6 and 2.11. This implies that $\tau_*(T)$ is closed.

We prove now the first identity in the lemma. Let θ be a smooth closed $(2k-2p)$-form such that $\mathrm{supp}(\theta)\cap\mathrm{supp}(\tau_*(T))$ is compact. We have seen that $\tau^*(\theta)$ is a closed current. It is enough to check that $\{\tau_*(T)\}\smile\{\theta\} = \{T\}\smile\{\tau^*(\theta)\}$. Since τ is smooth outside V, by definition of $\tau_*(T)$, we have

$$\{\tau_*(T)\}\smile\{\theta\} = \langle\tau_*(T),\theta\rangle = \langle T, \tau^*(\theta)\rangle_{W\setminus V}.$$

By de Rham's regularization theorem, there exist a sequence of smooth closed $(2k-2p)$-forms θ_n supported in a fixed open subset W' of W such that $\theta_n\to\tau^*(\theta)$ and $W'\cap\mathrm{supp}(T)$ is relatively compact in W. Moreover, since $\tau^*(\theta)$ is smooth outside V and bounded on W, the forms θ_n are bounded uniformly on n and converge locally uniformly to $\tau^*(\theta)$ on $W\setminus V$. Finally, since T has no mass on V, we have

$$\{T\}\smile\{\tau^*(\theta)\} = \lim_{n\to\infty}\{T\}\smile\{\theta_n\} = \lim_{n\to\infty}\langle T,\theta_n\rangle = \langle T,\tau^*(\theta)\rangle_{W\setminus V}.$$

Hence, $\{\tau_*(T)\} = \tau_*\{T\}$.

Consider the last identity in the lemma. Using the previous identity, it suffices to prove that $(\tau_{|V})_*(c_{|V}) = \tau_*(c)_{|\widetilde{V}}$ for any class $c\in H^*_V(W,\mathbb{C})$. We can assume

that c is represented by a smooth real manifold Y which intersects V transversally. We can also reduce W and $\widetilde{W}$ in order to assume that there is a projection $\widetilde{\Pi}:\widetilde{W}\to\widetilde{V}$ which defines a smooth fibration whose fibers are diffeomorphic to a ball. We deduce from the discussion before the lemma that

$$\tau_*(c)\smile\{\widetilde{V}\}=\tau_*(c)\smile\tau_*\{V\}=\tau_*(c\smile\{V\})=\tau_*\{Y\cap V\}=\{\tau(Y\cap V)\}.$$

Observe that $\tau_*(c)_{|\widetilde{V}}$ is the image of the class $\tau_*(c)\smile\{\widetilde{V}\}$ by the natural morphism $\widetilde{\Pi}_*:H^*_c(\widetilde{W},\mathbb{C})\to H^*_c(\widetilde{V},\mathbb{C})$. The last identities imply that $\tau_*(c)_{|\widetilde{V}}$ is equal to the class of $\tau(Y\cap V)$ in $H^*_c(\widetilde{V},\mathbb{C})$ which is equal to $(\tau_{|V})_*(c_{|V})$ since $c_{|V}$ is represented by $Y\cap V$. This completes the proof of the lemma. □

3 Currents on projective fibrations

In this section we discuss positive closed currents on fibrations over a complex manifold with projective spaces as fibers. These currents will appear as a kind of derivative in the normal direction, along a submanifold, of a positive closed current on a Kähler manifold.

Let V be a Kähler manifold of dimension l, not necessarily compact, and let ω_V be a Kähler form on V. Let E be a holomorphic vector bundle of rank r over V and denote by $\mathbb{P}(E)$ its projectivization. The complex manifold $\mathbb{P}(E)$ is of dimension $l+r-1$. Denote by $\pi:\mathbb{P}(E)\to V$ the canonical projection. The map π defines a regular fibration over V with $\mathbb{P}^{r-1}$ fibers.

Consider a Hermitian metric $\|\cdot\|$ on E and denote by $\omega_{\mathbb{P}(E)}$ the closed $(1,1)$-form on $\mathbb{P}(E)$ induced by $dd^c\log\|v\|$ with $v\in E$. The restriction of $\omega_{\mathbb{P}(E)}$ to each fiber of $\mathbb{P}(E)$ is the Fubini-Study form on this fiber. So $\omega_{\mathbb{P}(E)}$ is strictly positive in the fiber direction. It follows that given an open set $V_0\Subset V$ there is a constant $c>0$ large enough such that $c\pi^*(\omega_V)+\omega_{\mathbb{P}(E)}$ is positive on $\pi^{-1}(V_0)$ and defines a Kähler metric there.

Definition 3.1. Let S be a non-zero positive closed (p,p)-current on $\mathbb{P}(E)$. Let V_0 be an open subset of V. We call *horizontal dimension* (or *h-dimension* for short) of S over V_0 the largest integer j such that $S\wedge\pi^*(\omega_V^j)\neq 0$ on $\pi^{-1}(V_0)$. If this dimension is 0, we say that S is *vertical* over V_0. *The h-dimension of* S is its h-dimension over V. By convention, if $S=0$ on $\pi^{-1}(V_0)$ then the h-dimension of S over V_0 is $\max(l-p,0)$.

Note that if s is the h-dimension of S then $\max(l-p,0)\leq s\leq\min(l+r-p-1,l)$. Note also that the positivity of S and the strict positivity of ω_V imply that the definition does not depend on the choice of ω_V. In fact, we have the following general result.

Lemma 3.2. *Let p and q be fixed positive integers. Then the h-dimension of S over V_0 is strictly smaller than* $\max(p,q)$ *if and only if $S\wedge\pi^*(\theta)=0$ for every continuous (or smooth) (p,q)-form θ on V_0.*

Proof. Define $j:=\max(p,q)$. Observe that ω_V^j can be written as a finite combination of $\gamma\wedge\theta$ where γ is a smooth $(j-p,j-q)$-form and θ is a smooth (p,q)-form. Therefore, the sufficiency of the condition is clear. Assume now that the h-dimension of S over V_0 is strictly smaller than j. We prove that $S\wedge\pi^*(\theta)=0$.

Consider first the case where $p=q=j$. Since the problem is local on V_0, we can assume that θ has compact support in V_0. Moreover, we can write it as a finite combination of positive forms with compact support. So we can assume that θ is positive and $\theta\leq\omega_V^j$. Therefore, we have $0\leq S\wedge\pi^*(\theta)\leq S\wedge\pi^*(\omega_V^j)=0$. It follows that $S\wedge\pi^*(\theta)=0$.

Consider now the case where $(p,q)=(j,j-r)$ with $1\leq r\leq j$. The remaining case can be treated in the same way. Observe that θ can be written as a finite sum of forms of type $\gamma\wedge\beta$ where γ is a continuous $(r,0)$-form and β is a positive continuous $(j-r,j-r)$-form. So we can assume that $\theta=\gamma\wedge\beta$. Consider a test smooth form θ' of appropriate bidegree with compact support in $\pi^{-1}(V_0)$. We have to check that $\langle S\wedge\pi^*(\gamma\wedge\beta),\theta'\rangle=0$.

As above, we can assume that $\theta'=\gamma'\wedge\beta'$ with γ' of bidegree $(0,r)$ and β' positive of appropriate bidegree. From Cauchy-Schwarz's inequality, we have

$$\big|\langle S\wedge\pi^*(\gamma\wedge\beta),\theta'\rangle\big|\leq\big|\langle S\wedge\pi^*(\gamma\wedge\overline{\gamma}\wedge\beta),\beta'\rangle\big|^{1/2}\big|\langle S\wedge\pi^*(\beta),\gamma'\wedge\overline{\gamma}'\wedge\beta'\rangle\big|^{1/2}.$$

The first factor in the right hand side vanishes according to the bidegree (j,j) case. The result follows. Note that the same proof holds for T weakly positive and for $(p,q)=(j,j),(j-1,j)$ or $(j,j-1)$. □

The following lemma describes the structure of vertical currents.

Lemma 3.3. *Let S be a positive closed (p,p)-current on $\mathbb{P}(E)$ as above. Assume that S is vertical over an open set V_0. Then there exist a unique positive measure μ on V_0 and for μ almost every x, a positive closed (p,p)-current S_x on $\mathbb{P}(E)$ supported by $\pi^{-1}(x)$ and cohomologous to a linear subspace there, such that*

$$S=\int S_x d\mu(x)\quad on\quad \pi^{-1}(V_0).$$

Moreover, μ depends linearly on S.

Proof. By Lemma 3.2, for any smooth function χ on U the current $(\chi\circ\pi)S$ is closed on $\pi^{-1}(V_0)$. Since the problem is local on V_0, multiplying S with a function $\chi\circ\pi$ with compact support permits to assume that S has support in $\pi^{-1}(K)$ for some compact subset K of V_0. Fix a neighbourhood $V_1\Subset V_0$ of K. We have seen that $\pi^{-1}(V_1)$ is a Kähler manifold. Fix a Kähler form on it.

The set of all positive closed (p,p)-currents of mass 1 on $\pi^{-1}(V_1)$ satisfying the above property is a convex compact set. Its extremal elements should be supported by a fiber. It follows from Choquet's representation theorem that there is a positive measure μ on V such that for μ-almost every x there is a positive closed (p,p)-current S_x on $\mathbb{P}(E)$ supported by $\pi^{-1}(x)$ such that

$$S = \int S_x d\mu(x).$$

We can multiply μ with a positive function $\lambda(x)$ and divide S_x by $\lambda(x)$ in order to have that S_x is cohomologous to a linear subspace in $\pi^{-1}(x)$. We check that μ is unique and depends linearly on S.

Fix a closed form Ω of bidegree $(l+r-1-p, l+r-1-p)$ on $\mathbb{P}(E)$ such that its restriction of to each fiber of π is cohomologous to a linear subspace in this fiber, e.g. a power of $\omega_{\mathbb{P}(E)}$. We have

$$S \wedge \Omega = \int (S_x \wedge \Omega) d\mu(x).$$

The intersection $S_x \wedge \Omega$ defines a measure with algebraic mass 1. It follows that $\mu = \pi_*(S \wedge \Omega)$. So μ is unique and depends linearly on S. This completes the proof of the lemma. □

The last lemma and the following one give the complete description of currents with minimal h-dimension, i.e. of h-dimension $\max(l-p, 0)$.

Lemma 3.4. *Let S be a positive closed (p,p)-current on $\mathbb{P}(E)$ with $p < l$. Assume that the h-dimension of S over V_0 is smaller or equal to $l-p$. Then there is a unique positive closed (p,p)-current S^h on V_0 such that $S = \pi^*(S^h)$ on $\pi^{-1}(V_0)$. In particular, the h-dimension of S over V_0 is equal to $l-p$.*

Proof. The lemma is clear for $S = 0$. So we can assume that $S \neq 0$. The uniqueness of S^h is also clear. We prove now the existence of S^h. Since this is a local problem, we can assume that V_0 is a small ball and $\pi^{-1}(E)$ can be identified with the product $V_0 \times \mathbb{P}^{r-1}$ where π is identified with the canonical projection on V_0.

Denote by $x = (x_1, \ldots, x_l)$ the complex coordinates on V_0. If $I = (i_1, \ldots, i_m)$ with $i_j \in \{1, \ldots, l\}$, define $dx_I := dx_{i_1} \wedge \ldots \wedge dx_{i_m}$ and $d\overline{x}_I := d\overline{x}_{i_1} \wedge \ldots \wedge d\overline{x}_{i_m}$. Lemma 3.2 implies that S can be written on $V_0 \times \mathbb{P}^{r-1}$ as

$$S = \sum_{|I|=|J|=p} R_{IJ} dx_I \wedge d\overline{x}_J,$$

where $R_{I,J}$ is a 0-current on $V_0 \times \mathbb{P}^{r-1}$.

Since S is closed, we deduce that dR_{IJ} is a combination of dx_i and $d\overline{x}_i$, i.e. R_{IJ} is constant along the fibers of π (to see this point, we can regularize S using

some convolution). Therefore, R_{IJ} is the pull-back of a 0-current on V_0. We deduce the existence of a current S^h on V_0 such that $S = \pi^*(S^h)$. It is clear that S^h should be a positive closed (p,p)-current. □

Proposition 3.5. *Let S be a positive closed (p,p)-current on $\mathbb{P}(E)$ as above and let s be the h-dimension of S over an open set V_0. Let Ω be a smooth closed form of bidegree $(l-s+r-1-p, l-s+r-1-p)$ on $\pi^{-1}(V_0)$ whose restriction to each fiber of π is cohomologous to a linear subspace in the fiber. Then the current $S^h := \pi_*(S \wedge \Omega)$ on V_0 is positive closed of bidegree $(l-s, l-s)$ with support $\pi(\mathrm{supp}(S)) \cap V_0$. Moreover, it does not depend on the choice of Ω.*

Proof. It is clear that S^h is a closed $(l-s, l-s)$-current. Observe that there is a form Ω satisfying the hypothesis, e.g. a power of $\omega_{\mathbb{P}(E)}$. Since the problem is local, we can assume that V_0 is strictly contained in V and therefore $\pi^{-1}(V_0)$ is a Kähler manifold. In particular, we obtain a strictly positive form Ω by taking a linear combination of a power of $\omega_{\mathbb{P}(E)}$ and a power of $\pi^*(\omega_V)$. If we choose Ω strictly positive, we obtain a positive current S^h with support $\pi(\mathrm{supp}(S))$.

It remains to prove that S^h does not depend on the choice of Ω. By Lemma 3.2, if α is a positive closed (s,s)-form on V then $S \wedge \pi^*(\alpha)$ is a vertical positive closed $(p+s, p+s)$-current. It follows from this lemma that $S \wedge \pi^*(\chi\alpha)$ is a vertical positive closed $(p+s, p+s)$-current for any positive function χ on V. By Lemma 3.3, we can associate to this current a measure μ which depends linearly on $\chi\alpha$. We have seen in the proof of that lemma that μ is equal to $\pi_*(S \wedge \pi^*(\chi\alpha) \wedge \Omega) = S^h \wedge \chi\alpha$ and does not depend on the choice of Ω. Since any (s,s)-form β can be written as a finite combination of forms of type $\chi\alpha$, the measure $S^h \wedge \beta$ does not depend on the choice of Ω. We deduce that S^h is independent of the choice of Ω. □

Definition 3.6. With the notation as in Proposition 3.5, we say that S^h is *the shadow of S on V_0. The shadow of S* is its shadow on V.

Denote by $-h_{\mathbb{P}(E)}$ the tautological class on $\mathbb{P}(E)$ which is the Chern class of the tautological line bundle $O_{\mathbb{P}(E)}(-1)$ over $\mathbb{P}(E)$. With the notation as in the beginning of the section, $h_{\mathbb{P}(E)}$ is the class of $\omega_{\mathbb{P}(E)}$. Recall that the cohomology ring $\oplus H^*_c(\mathbb{P}(E), \mathbb{C})$ is a free $\oplus H^*_c(V, \mathbb{C})$-module generated by the classes $1, h_{\mathbb{P}(E)}, \ldots h^{r-1}_{\mathbb{P}(E)}$, see e.g. Bott-Tu [6] or Voisin [40]. This is a consequence of Leray's theorem and can also be deduced from a similar property for de Rham cohomology without compact support via the Poincaré duality. So if c is a class in $H^{2p}_c(\mathbb{P}(E), \mathbb{C})$, we can write it, in a unique way as

$$c = \sum_{j=\max(0,l-p)}^{\min(l,l+r-1-p)} \pi^*(\kappa_j(c)) \smile h^{j-l+p}_{\mathbb{P}(E)},$$

where $\kappa_j(c)$ is a class in $H^{2l-2j}_c(V, \mathbb{C})$. If c is the class of a closed (p,p)-current S with compact support on $\mathbb{P}(E)$, we write $\kappa_j(S) := \kappa_j(c)$.

Definition 3.7. The maximal j such that $\kappa_j(c) \neq 0$ is called *the horizontal dimension* (or *h-dimension* for short) of the class c. If $c = 0$, by convention, the h-dimension of c is $\max(l - p, 0)$.

The following lemma shows that the h-dimension of a positive closed current with compact support depends only on its cohomology class. Recall that the class c is said to be *pseudo-effective* if it contains a positive closed current.

Lemma 3.8. *Let S be a positive closed (p, p)-current with compact support in $\mathbb{P}(E)$. Then the h-dimension of S is equal to the h-dimension of $\{S\}$. Moreover, if S^h is the shadow of S and s is the h-dimension of S then S^h belongs to the class $\kappa_s(S)$. In particular, $\kappa_s(S)$ is a pseudo-effective class; if $S \neq 0$ and if $\pi(\mathrm{supp}(S))$ does not support a positive closed $(l-j, l-j)$-current for some $j \geq 1$, then $s < j$.*

Proof. Let θ be a smooth closed $2j$-form on V with $j > s$. By Lemma 3.2, we have $S \wedge \pi^*(\theta) = 0$. It follows from the above uniqueness of the decomposition of $H^{2p}_c(\mathbb{P}(E), \mathbb{C})$ that $\kappa_0(S \wedge \pi^*(\theta)) = \kappa_j(S) \smile \{\theta\}$. We deduce that $\kappa_j(S) \smile \{\theta\} = 0$ for every θ. Hence, by Poincaré's duality, $\kappa_j(S) = 0$ and the h-dimension of $\{S\}$ is at most equal to s. In order to complete the proof of the lemma, we have to check that S^h belongs to $\kappa_s(S)$.

For this purpose, it is enough to prove that the measure $S^h \wedge \theta$ belongs to $\kappa_0(S \wedge \pi^*(\theta))$ for any smooth closed $2s$-form θ on V. We show this property for every smooth $(q, 2s - q)$-form θ not necessarily closed. If $q \neq s$, we have $S^h \wedge \theta = 0$ for bidegree reasons and $S \wedge \pi^*(\theta) = 0$ according to Lemma 3.2. So we can assume that $q = s$. The form θ can be written as a combination of positive forms. Therefore, we can suppose that θ is positive. Replacing S with $S \wedge \pi^*(\theta)$ (which is closed according to Lemma 3.2) allows us to assume that $s = 0$, i.e. S is a vertical current, and θ is the constant function 1. So S^h coincides with the measure μ given in Lemma 3.3. Using the decomposition given in that lemma, we reduce the problem to the case where μ is a Dirac mass. We can then check the property without difficulty.

Note that when S is only weakly positive and V is compact, it is enough to consider θ of bidegree (j, j) or (s, s). So Lemma 3.2 applies in this case and gives the same result. □

We now introduce the notion of V-conic currents. They will be used in order to describe the tangent to a positive closed current on a complex manifold along a submanifold V. Let E be a holomorphic vector bundle of rank r over a Kähler manifold V of dimension l as above. We do not assume that V is compact and we identify it with the zero section.

The projectivisation $\mathbb{P}(E \oplus \mathbb{C})$ of the vector bundle $E \oplus \mathbb{C}$ is a natural compactification of E. Here $\mathbb{C}$ denotes the trivial line bundle over V. For simplicity, define $\overline{E} := \mathbb{P}(E \oplus \mathbb{C})$ and $\pi_0 : \overline{E} \to V$ the canonical projection. If V_0 is an open subset relatively compact in V, as we have seen above, $\pi_0^{-1}(V_0)$ is a Kähler

manifold. The action of the multiplicative group $\mathbb{C}^*$ on E extends naturally to $\overline{E}$.

Definition 3.9. A positive closed (p,p)-current S on E is V*-conic* if it is invariant under the action of $\mathbb{C}^*$.

We will see in Proposition 3.10 below that such a current, extended by 0 on the hypersurface at infinity $H_\infty := \overline{E} \setminus E$, is a positive closed current on $\overline{E}$ that we still denote S. Note that any current supported by V is V-conic.

Let $\pi_\infty : \overline{E} \setminus V \to H_\infty$ be the central projection on the hypersurface at infinity. We can also identify H_∞ with $\mathbb{P}(E)$ and the restriction of π_0 to H_∞ with $\pi : \mathbb{P}(E) \to V$. We have the following characterization of V-conic currents.

Proposition 3.10. *Let S be a V-conic positive closed (p,p)-current as above. Assume that* $\mathrm{supp}(S) \cap V$ *is compact. Then, there is a unique positive closed (p,p)-current S_∞ on $H_\infty \simeq \mathbb{P}(E)$ and a unique positive closed (p,p)-current S_0 on $\overline{E}$ with support in V such that*

$$S = \pi_\infty^*(S_\infty) + S_0.$$

In particular, S extends by 0 *to a positive closed current on $\overline{E}$ that we still denote S. Moreover, the intersection $S \wedge [H_\infty]$ is well-defined and is equal to S_∞. The currents S_∞, S_0 have compact supports and S_0 is the restriction of S to V.*

Note that $\pi_\infty^*(S_\infty)$ is well-defined on $\overline{E} \setminus V$ since π_∞ is a submersion there. The following lemma shows that this current extends by 0 to a positive closed (p,p)-current on $\overline{E}$ and we keep the same notation for the extended current. We already obtain here the uniqueness of S_∞. The last assertion in the proposition is also clear. Here, by restriction of S to V, we means the multiplication of S with the characteristic function of V. One should distinguish it with the intersection of S with $[V]$.

Lemma 3.11. *Let R be a current of order* 0 *with compact support on $H_\infty \simeq \mathbb{P}(E)$. Then the current $\pi_\infty^*(R)$ has finite mass near V. We still denote by $\pi_\infty^*(R)$ its extension by* 0 *through V. The operator $R \mapsto \pi_\infty^*(R)$ is continuous. If R is closed then $\pi_\infty^*(R)$ is closed and we have $\pi_\infty^*\{R\} = \{\pi_\infty^*(R)\}$.*

Proof. Let $\sigma_E : \widehat{\overline{E}} \to \overline{E}$ be the blow-up of $\overline{E}$ along V. In order to simplify the notation, we identify $\sigma_E^{-1}(H_\infty)$ with H_∞. The map π_∞ lifts naturally to a **holomorphic** map $\widehat{\pi}_\infty : \widehat{\overline{E}} \to H_\infty$ which defines a regular fibration with $\mathbb{P}^1$ fibers over H_∞. So the current $\widehat{\pi}_\infty^*(R)$ is well-defined and of order 0 and has no mass on $\sigma_E^{-1}(V)$. Therefore, $(\sigma_E)_*\widehat{\pi}_\infty^*(R)$ is a current of order 0 having no mass on V. It is equal to $\pi_\infty^*(R)$ outside V and depends continuously on R. The first and second assertions in the lemma follow.

For the last assertion, assume that R is closed. Clearly, $\pi_\infty^*(R)$ is closed. It remains to check the identity in the lemma. Since $\pi_\infty^*(R)$ depends continuously on

R, by de Rham's regularization theorem, it is enough to consider the case where R is a smooth form. Recall that the operator $\pi_\infty^*: H_c^*(H_\infty, \mathbb{C}) \to H_c^*(\mathbb{P}(E), \mathbb{C})$ is defined by $\pi_\infty^*\{R\} := \{(\sigma_E)_* \widehat{\pi}_\infty^*(R)\}$ for R smooth and closed. Note that even when R is smooth $(\sigma_E)_* \widehat{\pi}_\infty^*(R)$ should be considered as a current. The key point here is that de Rham cohomology groups can be defined using smooth forms or currents. The definition of the action of π_∞^* on cohomology is in fact valid for more general meromorphic maps. The last identity in the lemma is clear for smooth forms R. $\square$

We also need the following lemma. Assume that V is a submanifold of a Kähler manifold X of dimension k. We use the notations introduced in Section 2. Recall that $\sigma: \widehat{X} \to X$ is the blow-up of X along V and $\widehat{V} := \sigma^{-1}(V)$.

Lemma 3.12. *Let T be a positive closed (p,p)-current on X with support in a fixed open set W_1 of X such that $W_1 \cap V \Subset V$. Then, for every open sets $U \Subset U' \Subset X$ containing $W_1 \cap V$, the mass of $(\sigma_{|\widehat{X}\setminus\widehat{V}})^*(T)$ on $\sigma^{-1}(U) \setminus \widehat{V}$ is bounded by $c\|T\|_{U'}$ for some constant $c > 0$ independent of T. In particular, $(\sigma_{|\widehat{X}\setminus\widehat{V}})^*(T)$ extends by 0 to a positive closed (p,p)-current on $\widehat{X}$ that we denote by $\sigma^\diamond(T)$.*

Proof. The second assertion is a consequence of the first one and of an extension theorem by Skoda [38]. Let $\widehat{\omega}$ be a Kähler form on $\widehat{W_1} := \sigma^{-1}(W_1)$, see also Section 2. Observe that $R := \sigma_*(\widehat{\omega})$ is a positive closed $(1,1)$-current which is smooth outside V and has no mass on V. The mass of $(\sigma_{|\widehat{X}\setminus\widehat{V}})^*(T)$ on $\sigma^{-1}(U) \setminus \widehat{V}$ is equal to the mass of the measure $T \wedge R^{k-p}$ on $U \setminus V$. We have to bound the last quantity.

Let φ be the quasi-psh function on W_1 introduced in Section 2 such that $dd^c\varphi + c_0\omega \geq R$. This function is smooth outside V. Define for $M > 0$ large enough $\varphi_M := \max(\varphi, -M)$ and $\omega_M := c_0\omega + dd^c\varphi_M$. It is not difficult to see that ω_M is a positive closed current which is larger than R on the open set $\{\varphi > -M\}$. When $M \to \infty$, this open set increases to $W_1 \setminus V$. Therefore, it is sufficient to bound the mass of $T \wedge \omega_M^{k-p}$ on U by a quantity which is independent of M for M large enough.

Observe that the positive measure $T \wedge \omega_M^{k-p}$ is well-defined because ω_M has continuous local potentials. Moreover, for M large enough, $\varphi_M = \varphi$ on $W_1 \cap U' \setminus U$. It follows from Stokes' formula that the mass of $T \wedge \omega_M^{k-p}$ on U does not depend on M. Fix an M large enough. The classical Chern-Levine-Nirenberg's inequality implies that this mass is bounded by a constant times $\|T\|_{U'}$, see Chern-Levine-Nirenberg [7] and Demailly [10]. The lemma follows. $\square$

Note that the result can be generalized for maps between manifolds of different dimensions, see also [19]. In this paper, we only need the version stated above.

Definition 3.13. With the notations as in Lemma 3.12, we call $\sigma^\diamond(T)$ *the strict transform* of T by σ.

In general, $\sigma^*\{T\}$ is not equal to $\{\sigma^\diamond(T)\}$. The missing part is described in the following lemma.

Lemma 3.14. *With the notations of Lemma 3.12, there is a class $e(T)$ in $H^{2p-2}_c(\widehat{V},\mathbb{C})$ such that for any neighbourhood $\widehat{W}$ of $\widehat{V}$ the class $\sigma^*\{T\}-\{\sigma^\diamond(T)\}$ in $H^{2p}_{\widehat{V}}(\widehat{W},\mathbb{C})$ is equal to the canonical image of $e(T)$ in this cohomology group.*

Proof. Choose a neighbourhood $\widehat{W}'$ of $\widehat{V}$ which is $\widehat{V}$-contractile, i.e. there is a smooth projection $\Pi:\widehat{W}'\to\widehat{V}$ which defines a fibration with connected and simply connected fibers. By de Rham's regularization theorem, there is a current T' with support in W_1 smooth near $\widehat{V}$ and equal to T outside a compact set in $W':=\sigma(\widehat{W}')$ such that the class of $T-T'$ in $H^{2p}_c(W',\mathbb{C})$ vanishes. Define $e(T)$ as the class of the current $\Pi_*(\sigma^*(T')-\sigma^\diamond(T))$. It is clear that the property in the lemma is true for $\widehat{W}'$. Observe also that the $e(T)$ does not depend on the choice of T'.

Consider now an arbitrary open set $\widehat{W}$ as in the lemma. Choose a neighbourhood $\widehat{W}''\subset\widehat{W}\cap\widehat{W}'$ of $\widehat{V}$ such that Π restricted to $\widehat{W}''$ defines a fibration with connected and simply connected fibers. Since $e(T)$ does not depend on the above choice of T', we can choose a T' such that $T-T'$ is supported by $W'':=\sigma(\widehat{W}'')$ and its class in $H^*_c(W'',\mathbb{C})$ vanishes. As above, we see that the property in the lemma holds for $\widehat{W}$. □

Proof of Proposition 3.10. It is well-know that we can decompose S in a unique way into a sum of two positive closed (p,p)-currents $S=S'+S_0$ with S_0 supported by V and S' without mass on V, see e.g. Demailly [10] and Skoda [38]. For simplicity, we replace S with S' in order to assume that $S_0=0$ and S has no mass on V.

Consider first the case where V is a hypersurface. So π_∞ extends to a holomorphic map on $\overline{E}$ and defines a regular fibration over H_∞ with $\mathbb{P}^1$ fibers. Locally, we can identify this fibration with the product $B\times\mathbb{P}^1$ where B is a ball in $\mathbb{C}^{k-1}$. The hypersurface H_∞ is identified with $B\times\{\infty\}$ and the map π_∞ is identified with the canonical projection on B. The hypersurface V is identified with $B\times\{0\}$.

Since S is invariant under the action of $\mathbb{C}^*$, we can write on $B\times\mathbb{C}^*$ using the natural coordinates (z,t)

$$S=S_1(z)\wedge\frac{idt\wedge d\bar t}{t^2}+S_2(z)\wedge\frac{dt}{t}+\overline{S_2(z)}\wedge\frac{d\bar t}{\bar t}+S_3(z),$$

where the S_i are currents of order 0 and of the appropriate bidegree which do not depend on t. Since S has finite mass near $B\times\{0\}$, the first term vanishes. Then the positivity of S implies that the next two terms vanish. This implies the proposition for the hypersurface case with S_∞ such that $S_3=\pi^*_\infty(S_\infty)$.

Consider now the general case. We use the notation introduced in Lemma 3.11. Let $\widehat{S}$ be the strict transform of S by σ_E (we use here the hypothesis on the

support of S). The action of $\mathbb{C}^*$ on E extends to $\overline{E}$ and can be lifted to $\widehat{\overline{E}}$. The current $\widehat{S}$ is still invariant under this action. We can apply the hypersurface case considered above to the current $\widehat{S}$ and to the exceptional hypersurface $\sigma_E^{-1}(V)$. As in Lemma 3.11, for simplicity, we identify $\sigma_E^{-1}(H_\infty)$ with H_∞. So we can write $\widehat{S} = \widehat{\pi}_\infty^*(S_\infty)$ with a positive closed (p,p)-current S_∞ on H_∞. Since these currents have no mass on V, we deduce that $S = \pi_\infty^*(S_\infty)$. This completes the proof of the proposition. □

Let S be a V-conic current as above with compact support in $\overline{E}$. Let $-h_{\overline{E}}$ denote the tautological class of $\overline{E} = \mathbb{P}(E \otimes \mathbb{C})$. By Leray's theorem, we can decompose the class of S as

$$\{S\} = \sum_{j=\max(0,l-p)}^{\min(l,l+r-p)} \pi_0^*(\kappa_j(S)) \smile h_{\overline{E}}^{j-l+p}$$

where $\kappa_j(S)$ is a class in $H_c^{2l-2j}(V,\mathbb{C})$ with $\kappa_j(S) = 0$ when j does not satisfies the inequalities $\max(0, l-p) \leq j \leq \min(l, l+r-p)$. We have the following lemma

Lemma 3.15. *Let S, S_∞ and S_0 be as in Proposition 3.10. Then*

$$\kappa_{l+r-p}(S) = \{S_0\} \quad \text{and} \quad \kappa_j(S) = \kappa_j(S_\infty) \quad \text{for} \quad j < l+r-p.$$

In particular, if s is the h-dimension of S_∞, then $\kappa_j(S) = 0$ for $j > s$ except possibly for $j = l+r-p$ and $\kappa_s(S)$ contains the shadow of S_∞.

Proof. Observe that the second assertion is a consequence of the first one and of Lemma 3.8. Recall that for simplicity we identify H_∞ with $\mathbb{P}(E)$ and $\overline{E}$ with $\mathbb{P}(E \oplus \mathbb{C})$. The map π_∞ is induced by the canonical projection from $E \oplus \mathbb{C}$ to E. The pull-back of a Hermitian metric on E gives a singular Hermitian metric on $E \oplus \mathbb{C}$. We deduce that $\pi_\infty^*(h_{\mathbb{P}(E)}) = h_{\overline{E}}$. Using the blow-up as in Lemma 3.11, we obtain easily that

$$\pi_\infty^*(h_{\mathbb{P}(E)}^m) = h_{\overline{E}}^m \quad \text{for} \quad m < r \qquad \text{and} \qquad h_{\overline{E}}^r = [\pi_\infty^*(h_{\mathbb{P}(E)})]^r = \{V\}.$$

Therefore, using that π_0 is identified with $\pi \circ \pi_\infty$, we get

$$\begin{aligned} \{\pi_\infty^*(S_\infty)\} &= \sum_{j=\max(0,l-p)}^{\min(l,l+r-1-p)} \pi_\infty^* \pi^*(\kappa_j(S_\infty)) \smile \pi_\infty^*(h_{\mathbb{P}(E)}^{j-l+p}) \\ &= \sum_{j=\max(0,l-p)}^{\min(l,l+r-1-p)} \pi_0^*(\kappa_j(S_\infty)) \smile h_{\overline{E}}^{j-l+p} \end{aligned}$$

and

$$\{S_0\} = \pi_0^*\{S_0\} \smile \{V\} = \pi_0^*\{S_0\} \smile h_{\overline{E}}^r.$$

Then, the lemma follows from the identity $S = \pi_\infty^*(S_\infty) + S_0$ and the uniqueness of the above decompositions. □

In the following lemma, we can use any fixed Hermitian metric on $\overline{E}$.

Lemma 3.16. *Let K be a fixed compact subset of V and let U be a fixed neighbourhood of K in E. If S is a V-conic (p,p)-current with support in $\pi_0^{-1}(K)$, then $\|S\| \leq c\|S\|_U$ for some constant $c > 0$ independent of S.*

Proof. If the lemma were wrong, there would be a sequence of V-conic (p,p)-currents (S_n) supported by $\pi_0^{-1}(K)$ such that $\|S_n\| \geq n\|S_n\|_U$. We can divide each S_n by its mass in order to assume that $\|S_n\| = 1$. Extracting a subsequence allows to assume that S_n converges to a V-conic current S of mass 1 which vanishes on U. Since this current is V-conic, it vanishes on E. This is a contradiction. $\square$

We come back to the case where V is a submanifold of dimension l of a Kähler manifold X of dimension k. Let $\sigma : \widehat{X} \to X$ and $\widehat{V} := \sigma^{-1}(V)$ be as above. Denote by E the normal vector bundle to V in X. Then the exceptional hypersurface $\widehat{V}$ of $\widehat{X}$ is canonically identified with $\mathbb{P}(E)$. So we can identify the restriction of σ to $\widehat{V}$ with $\pi : \mathbb{P}(E) \to V$. We will need the following lemma.

Lemma 3.17. *Let S be a positive closed (p,p)-current on $\widehat{X}$ with compact support in $\widehat{V} = \mathbb{P}(E)$ and with $p \geq 1$. Let s be the h-dimension of S. Assume that s is strictly smaller than the complex dimension $k-p$ of S. Let $\{S\}'$ denote the class of S in $H^{2p}_{\widehat{V}}(\widehat{X},\mathbb{C})$. Then $\kappa_j(\{S\}'_{|\widehat{V}}) = 0$ if $j > s$ and $-\kappa_s(\{S\}'_{|\widehat{V}})$ contains the shadow of S on V. In particular, the class $-\kappa_s(\{S\}'_{|\widehat{V}})$ is pseudo-effective.*

Proof. Using a diffeomorphism from a neighbourhood of $\widehat{V}$ in $\widehat{X}$ to a neighbourhood of $\widehat{V}$ in $\widehat{\overline{E}}$ which is identity on $\widehat{V}$, we can reduce the problem to the case where $X = \overline{E}$ and $\widehat{X} = \widehat{\overline{E}}$. Let $\widehat{\pi}_0 : \widehat{\overline{E}} \to \widehat{V}$ be the canonical projection. It defines a fibration with $\mathbb{P}^1$ fibers over $\widehat{V}$.

We can identify S with the intersection of $\widehat{\pi}_0^*(S)$ with $[\widehat{V}]$. Let $\{S\}$ denote the class of S in $H^{2p-2}_c(\widehat{V},\mathbb{C})$. We have

$$\{S\}'_{|\widehat{V}} = (\widehat{\pi}_0^*\{S\} \smile [\widehat{V}])_{|\widehat{V}} = \{S\} \smile [\widehat{V}]_{|\widehat{V}} = -\{S\} \smile h_{\mathbb{P}(E)}.$$

Finally, since s is strictly smaller than the complex dimension of S, we deduce from the definition of $\kappa_j(\cdot)$ that $-\kappa_j(\{S\}'_{|\widehat{V}}) = \kappa_j(\{S\})$. The lemma follows. $\square$

4 Tangent cones for positive closed currents

In this section, we introduce the tangent cones, along a submanifold, of a positive closed current on a Kähler manifold. We refer the reader to Siu [37] for the case where the submanifold is just a point, i.e. the case of Lelong number.

Let X be a Kähler manifold of dimension k. Let V be a submanifold of dimension l. Let T be a positive closed (p,p)-current on X such that $\mathrm{supp}(T)\cap V$

is compact. The later condition is satisfied when V or X is already compact. We want to define tangent currents to T along V. They are V-conic currents on $\overline{E}$ where $E := N_{V|X}$ is the normal vector bundle to V in X. We need a special class of homeomorphisms from neighbourhoods of V in X to neighbourhoods of V in E which are in some sense close to holomorphic maps near V.

Consider a point a in V. If $\mathrm{Tan}_a X$ and $\mathrm{Tan}_a V$ denote respectively the tangent spaces of X and of V at a, the fiber E_a of E over a is canonically identified with the quotient space $\mathrm{Tan}_a X/\mathrm{Tan}_a V$. Let $x = (x', x'')$ with $x' = (x_1, \ldots, x_l)$ and $x'' = (x_{l+1}, \ldots, x_k)$ be a local holomorphic coordinate system that identifies a chart of X to the polydisc $2\mathbb{D}^k$ in $\mathbb{C}^k$ such that V is defined by the equation $x'' = 0$ in this polydisc. In these local coordinates, the bundle E is canonically identified over $V \cap 2\mathbb{D}^k$ with the trivial bundle $(V \cap 2\mathbb{D}^k) \times \mathbb{C}^{k-l}$ which is an open subset of $\mathbb{C}^k$ containing $2\mathbb{D}^k$.

Let V_0 be an open subset of V. Let τ be a bi-Lipschitz map from a neighbourhood of V_0 in X to a neighbourhood of V_0 in E. We assume that τ is equal to identity on V_0 and is smooth outside V_0.

Definition 4.1. We say that τ is *admissible* (resp. *almost-admissible*) if in any local holomorphic coordinate system as above τ is admissible (resp. almost-admissible) in the sense of Definition 2.14 (resp. Definition 2.18).

Note that if τ is smooth and admissible its differential at any point of V is $\mathbb{C}$-linear and induces an endomorphism of E which is equal to identity. If V_0 is small enough we can find τ admissible and holomorphic. In general, we have the following lemma.

Lemma 4.2. *There is a smooth admissible map for $V_0 = V$.*

Proof. Consider a Hermitian metric on X. It induces a Hermitian metric on the tangent bundle of X. Denote by F the restriction of this tangent bundle to V. The tangent bundle of V is identified with a vector sub-bundle F' of F and E is identified with F/F'. Let F'' denote the orthogonal complement of F' in F. This is a vector bundle over V with complex fibers but in general it is **not a holomorphic** vector bundle.

The canonical projection $\tau_1 : F'' \to E$ is a smooth isomorphism between vector bundles and is $\mathbb{C}$-linear on each fiber. Let $\tau_2 : F'' \to X$ be the map induced by the exponential maps at the points of V. This map defines a smooth diffeomorphism between a neighbourhood of V in F'' and a neighbourhood of V in X. It is identity on V and its differential at each point of V is identity. Define $\tau := \tau_1 \circ \tau_2^{-1}$ on a small neighbourhood of V in X. This is a diffeomorphism between this neighbourhood and its image which is a neighbourhood of V in E.

In local coordinates $x = (x', x'')$ as above, we can find smooth matrix-functions $a(x)$, such that the fiber of F'' over a point $(x', 0)$ is the set of points $(x' + x''a(x), x'')$ with $x'' \in \mathbb{C}^{k-l}$. In a small neighbourhood of V, these affine spaces are pairwise disjoint. The map τ_1 sends $(x' + x''a(x), x'')$ to (x', x''). The map

τ_2^{-1} is smooth and tangent to identity at each point of V. So we have $\tau_2^{-1}(x) = x + O(\|x''\|^2)$ and $d\tau_2^{-1}(x) = dx + O^{**}(\|x''\|^2)$. So τ satisfies Definition 2.14. $\square$

In what follows, we often use the blow-up $\sigma : \widehat{X} \to X$ of X along V and the blow-up $\sigma_E : \widehat{\overline{E}} \to \overline{E}$ of $\overline{E}$ along V. We will show that admissible maps on X can be lifted to almost-admissible maps on $\widehat{X}$. However, in general, we loose the smoothness of these maps and they are only bi-Lipschitz. This is the motivation for Definition 2.14.

Observe that $\sigma_E^{-1}(V)$ can be canonically identified with $\mathbb{P}(E)$. So we also identify it with $\widehat{V}$. The restriction of σ_E to this hypersurface is identified with the restriction of σ to $\widehat{V}$ and with $\pi : \mathbb{P}(E) \to V$. For simplicity, we identify $\sigma_E^{-1}(H_\infty)$ with H_∞. The projections $\pi_0 : \overline{E} \to V$ and $\pi_\infty : \overline{E} \setminus V \to H_\infty$ lift to projections $\widehat{\pi}_0 : \widehat{\overline{E}} \to \widehat{V}$ and $\widehat{\pi}_\infty : \widehat{\overline{E}} \setminus \widehat{V} \to H_\infty$, that is, we have $\sigma_E \circ \widehat{\pi}_0 = \pi_0 \circ \sigma_E$ and $\widehat{\pi}_\infty = \pi_\infty \circ \sigma_E$. Finally, $\widehat{E} := \widehat{\overline{E}} \setminus H_\infty$ can be identified with the normal line bundle to $\widehat{V}$ in $\widehat{X}$ and $\widehat{\overline{E}}$ is its natural compactification.

Lemma 4.3. *Let τ be the smooth admissible map constructed in Lemma 4.2. Then there is a unique almost-admissible map $\widehat{\tau}$, from a neighbourhood of $\widehat{V}_0 := \sigma^{-1}(V_0)$ to a neighbourhood of $\widehat{V}_0$ in $\widehat{E}$ such that $\sigma_E \circ \widehat{\tau} = \tau \circ \sigma$.*

Proof. Since σ and σ_E are biholomorphic maps outside $\widehat{V}$, we necessarily have $\widehat{\tau} = \sigma_E^{-1} \circ \tau \circ \sigma$ outside $\widehat{V}$. By continuity, the map $\widehat{\tau}$ is unique if it exists. We will describe $\widehat{\tau}$ outside $\widehat{V}$ using local coordinates and we will see that it extends to an almost-admissible map.

Let $x = (x', x'')$ be as above where we identify a chart of X with $2\mathbb{D}^k$. The restriction of V to $2\mathbb{D}^k$ is given by the equation $x'' = 0$. The vector bundle E is identified over $V \cap 2\mathbb{D}^k$ with $2\mathbb{D}^l \times \mathbb{C}^{k-l}$. The map τ is described as in Definition 2.14. In these coordinates, we identify X with E and σ with σ_E over $2\mathbb{D}^k$. Consider the chart $\widehat{D}$ of $\sigma^{-1}(2\mathbb{D}^k)$ introduced in Section 2 with coordinates $z = (z_1, \ldots, z_k)$ such that $|z_j| < 2$ and

$$\sigma(z) = (z_1, \ldots, z_l, z_{l+1}z_k, \ldots, z_{k-1}z_k, z_k).$$

In these coordinates, $\widehat{V}$ is given by $z_k = 0$. We have

$$\sigma_E^{-1}(x) = \sigma^{-1}(x) = (x_1, \ldots, x_l, x_{l+1}x_k^{-1}, \ldots, x_{k-1}x_k^{-1}, x_k).$$

Define $z' := (z_1, \ldots, z_l)$ and $z^\# := (z_{l+1}, \ldots, z_{k-1})$. Using the local description of τ in the proof of Lemma 4.2, we can write the map $\widehat{\tau} := \sigma_E^{-1} \circ \tau \circ \sigma$ on $\widehat{D} \setminus \widehat{V}$ in coordinates z as

$$\widehat{\tau}(z) = \big(z' + O^*(|z_k|), z^\# + O^*(|z_k|), z_k + O^*(|z_k|^2)\big).$$

We see that $\widehat{\tau}$ extends continuously to a map on $\sigma^{-1}(\mathbb{D}^k)$ which is identity on $\widehat{V}$.

In the last identity, the function $O^*(|z_k|^2)$ is smooth and $O^*(|z_k|)$ is the product of z_k^{-1} with a smooth $O(|z_k|^2)$ function. Since smooth $O(|z_k|^2)$ functions can be written as a combination of $z_k^2, z_k\overline{z}_k$ and $\overline{z}_k^2$ with smooth coefficients, we see that

$$d\widehat{\tau}(z) = \big(dz' + O^{**}(|z_k|), dz^\# + O^{**}(|z_k|), dz_k + O^{**}(|z_k|^2)\big).$$

Hence, $\widehat{\tau}$ is almost-admissible. $\square$

Denote by A_λ the automorphism of $\overline{E}$ induced by the multiplication by $\lambda \in \mathbb{C}^*$. Let τ be an **almost-admissible** map as in Definition 4.1. Fix an open subset W_1 of X such that $W_1 \cap V \neq \varnothing$. Consider a positive closed (p,p)-current on X with support in W_1. We can decompose T as $T' + T_0$ where T', T_0 are positive closed currents, T' has no mass on V and T_0 is the restriction of T to V.

By Lemma 2.20, we can define a closed $2p$-current $\tau_*(T')$ of order 0 on a neighbourhood of V_0 in E with no mass on V. Since τ is identity on V, we define $\tau_*(T) = \tau_*(T') + T_0$. Define $T_\lambda := (A_\lambda)_*(\tau_*(T))$. This is a closed $2p$-current whose domain of definition converges to an open set containing $\pi_0^{-1}(V_0) \setminus H_\infty$.

Proposition 4.4. *Let U be an open set relatively compact in $\pi_0^{-1}(V_0) \setminus H_\infty$. Then for λ large enough, the current T_λ is defined on U and its mass on U is bounded by $c\|T\|$ for some constant c independent of λ and of T. Moreover, if (λ_n) is a sequence converging to infinity such that T_{λ_n} converges to a current S on $\pi_0^{-1}(V_0) \setminus H_\infty$, then S is a positive closed (p,p)-current independent of the choice of τ.*

Proof. We only consider λ large enough. So the current T_λ is closed and is defined on U. Since the problem is local with respect to V, we can assume that V_0 is equal to $\mathbb{D}^k \cap V$, where $\mathbb{D}^k$ is identified with a chart of X with local coordinates x as above.

Let R be a smooth $(2k-2p)$-form with compact support in U and with coefficients bounded by 1. We have $\langle T_\lambda, R\rangle = \langle T, \tau^*(A_\lambda)^*(R)\rangle$. By Remark 2.19, the last integral is bounded by a constant times $\|T\|$. It follows that the mass of T_λ on U is bounded by a constant times $\|T\|$. In particular, for any sequence (λ_n) converging to infinity, we can extract a subsequence (λ_{n_i}) such that $T_{\lambda_{n_i}}$ converges to a closed current on $\pi_0^{-1}(V_0) \setminus H_\infty$.

Let R' denote the component of bidegree $(k-p, k-p)$ of R. Define $R_\lambda := (A_\lambda)^*(R')$. By Remark 2.19, $\langle T_\lambda, R\rangle - \langle T, R_\lambda\rangle$ tends to 0 as λ tends to infinity. Since R_λ does not depend on the choice of τ, we deduce that S does not depend on the choice of τ. If R is of bidegree (q_1, q_2) with $(q_1,q_2) \neq (k-p,k-p)$ then $R' = 0$. It follows that $\langle S, R\rangle = 0$. Hence, S is a current of bidegree (p,p). When R is a weakly positive $(k-p,k-p)$-form, R_λ is also weakly positive and hence $\langle T, R_\lambda\rangle$ is positive. We deduce that $\langle S, R\rangle$ is positive. Therefore, S is a positive current. $\square$

Let τ be a global **almost-admissible** map as in Definition 4.1 for $V_0 = V$. We define as above $T_\lambda := (A_\lambda)_* \tau_*(T)$. By Proposition 4.4, the following notion of tangent current does not depend on the choice of τ.

Definition 4.5. A current S on E is said to be a *tangent current* to T along V if there is a sequence $\lambda_n \to \infty$ such that $S = \lim T_{\lambda_n}$. We also say that S is the tangent current associated to the sequence (λ_n).

Observe that if θ is a smooth positive closed (q,q)-form on X and if S is the tangent current to T along V associated to a sequence (λ_n) then $S \wedge \pi_0^*(\theta_{|V})$ is the tangent current to $T \wedge \theta$ along V associated to (λ_n).

Theorem 4.6. *Let X be a Kähler manifold and let V be a submanifold of X. Denote by E the normal vector bundle to V in X, $\overline{E}$ its natural compactification and $\pi_0 : \overline{E} \to V$ the canonical projection. Let W_1 be a fixed open subset of X such that $W_1 \cap V$ is relatively compact in X. If T is a positive closed (p,p)-current on X with support in W_1, then its tangent currents along V are V-conic supported by $\pi_0^{-1}(\mathrm{supp}(T) \cap V)$ and of mass bounded by $c\|T\|$ for some constant c independent of T. Moreover, these tangent currents belong to the same cohomology class in $H^{2p}_c(\overline{E}, \mathbb{R})$ and their restrictions to V are equal to the restriction of T to V.*

We need the following lemma where τ is **smooth admissible** and $\widehat{\tau}$ is given by Lemma 4.3.

Lemma 4.7. *Let S be the tangent current to T along V associated to a sequence (λ_n). Then the restriction of S to V is equal to the restriction of T to V. Let $\widehat{T}$ be the strict transform of T by $\sigma : \widehat{X} \to X$. Then $\widehat{T}$ admits a tangent current $\widehat{S}$ along $\widehat{V}$ associated to the same sequence (λ_n). Moreover, $\widehat{S}$ is the strict transform of S by $\sigma_E : \widehat{\overline{E}} \to \overline{E}$.*

Proof. If T is supported by V then $\widehat{T} = 0$ and $S = T$. The lemma is clear. So we can assume that T has no mass on V. We can replace (λ_n) by a subsequence in order to assume that $\widehat{T}$ admits a tangent current $\widehat{S}$ along $\widehat{V}$ associated to (λ_n). We have to check that it is the strict transform of S and that $(\sigma_E)_*(\widehat{S}) = S$. The last equality implies that S has no mass on V.

Denote by $\widehat{A}_\lambda$ the map on $\widehat{\overline{E}}$ induced by the multiplication by λ. It is the lift of A_λ to $\widehat{\overline{E}}$. Define also $\widehat{T}_\lambda := (\widehat{A}_\lambda)_* \widehat{\tau}_*(\widehat{T})$. Let R be a test smooth $(2k-2p)$-form with compact support in E. Define $\widehat{R} := \sigma_E^*(R)$. It is not difficult to see that $\langle \widehat{T}_\lambda, \widehat{R} \rangle = \langle T_\lambda, R \rangle$ because T_λ and $\widehat{T}_\lambda$ have no mass on V and $\widehat{V}$ respectively. It follows that $(\sigma_E)_*(\widehat{S}) = S$. Since σ_E is injective outside $\widehat{V}$, it remains to check that $\widehat{S}$ has no mass on $\widehat{V}$.

In order to simplify the notation, we consider the case where V is a hypersurface and T has no mass on V. We have to check that S has no mass on V. The result we obtain when applied to $\widehat{X}, \widehat{V}, \widehat{T}$ and $\widehat{S}$, gives the lemma. Multiplying

T with a strictly positive closed form allows us to reduce the problem to the case where T is of bidegree $(k-1,k-1)$, see the observation before Theorem 4.6.

We use local coordinates $x=(x',x_k)$ with $x'=(x_1,\ldots,x_{k-1})$ on a chart $\mathbb{D}^k$ as above. Let γ denote the restriction of $dd^c\|x'\|^2$ to $\mathbb{D}^k$. The mass of S on $V\cap\mathbb{D}^k$ is bounded by a constant times $\langle S,\gamma\rangle$. Arguing as in Proposition 4.4, we see that the last integral is bounded by

$$\limsup_{\lambda\to\infty}\langle T,(A_\lambda)^*(\gamma)\rangle=\limsup_{\lambda\to\infty}\langle T,dd^c\|x'\|^2\rangle_{\mathbb{D}^l\times\lambda^{-1}\mathbb{D}^{k-l}}=0$$

since T has no mass on V. It follows that S has no mass on V. The proof of the lemma is now complete. □

End of the proof of Theorem 4.6. The theorem is clear when T is supported by V. So we can assume that T has no mass on V. The last assertion is already obtained in Lemma 4.7. The mass estimate for tangent currents is a consequence of Proposition 4.4 and Lemma 3.16. The assertion on the supports of the tangent currents is also clear. We prove now that the tangent currents are V-conic and that they have the same cohomology class. By Lemma 4.7, we can assume that V is a hypersurface of X.

We use a chart $\mathbb{D}^k$ of X as above with local coordinates $x=(x_1,\ldots,x_k)$ such that $V\cap\mathbb{D}^k$ is given by the equation $x_k=0$. Let R be a smooth $(k-p,k-p)$-form with compact support in $\mathbb{D}^k$. With notations as above, we have seen in Proposition 4.4 that $\langle T_\lambda,R\rangle-\langle T,(A_\lambda)^*(R)\rangle$ converges to 0 as λ tends to infinity. We apply Lemma 2.13 to our situation. We also use the fact that T is closed and hence vanishes on exact test forms. We obtain for every fixed $t\in\mathbb{C}^*$ that $\langle T,(A_{t\lambda})^*(R)\rangle-\langle T,(A_\lambda)^*(R)\rangle$ converges to 0. It follows that $\langle T_{t\lambda},R\rangle-\langle T_\lambda,R\rangle$ tends to 0. Therefore, tangent currents to T along V are invariant under the action of $(A_t)_*$, i.e. they are V-conic currents.

We prove that the tangent currents to T have the same cohomology class. Let S be such a current. Fix also a small neighbourhood of V in E. It is not difficult to see that for λ large enough T_λ restricted to this neighbourhood is a closed current whose cohomology class does not depend on λ. It follows that $\{S\}\smile\{V\}$ does not depend on the choice of S. Since S is V-conic and V is a hypersurface, we deduce that the class of S is $H^*_c(\overline{E},\mathbb{C})$ does not depend on the choice of S. □

Let S be a tangent current to T along V. Denote by $\kappa^V(T)$ the class of S in $H^{2p}_c(\overline{E},\mathbb{C})$. We know that it does not depend on the choice of S.

Definition 4.8. We say that $\kappa^V(T)$ is the *total tangent class* of T along V. The h-dimension of $\kappa^V(T)$ is *the tangential h-dimension of T along V*. The *set of tangent directions* of T along V is the union of $\mathrm{supp}(S)$ for S varying on the set of all tangent currents. Its projection to V is the *tangent locus* of T along V.

If $-h_{\overline{E}}$ denotes the tautological class of $\overline{E}$, as in Section 3, we can write in a unique way

$$\kappa^V(T) = \sum_{j=\max(0,l-p)}^{\min(l,k-p)} \pi_0^*(\kappa_j^V(T)) \smile h_{\overline{E}}^{j-l+p},$$

where $\kappa_j^V(T)$ is a class in $H_c^{2l-2j}(V,\mathbb{C})$. By convention, $\kappa_j^V(T)$ is 0 when j does not satisfy the inequalities $\max(0,l-p) \leq j \leq \min(l,k-p)$. With notations as in Section 3, we have $\kappa_j^V(T) = \kappa_j(S)$ if S is a tangent current to T along V.

Remarks 4.9. Let θ be a smooth positive closed (q,q)-form on X with $q \leq k-p$. Let S be the tangent current to T associated to a sequence (λ_n). Then $S \wedge \pi_0^*(\theta_{|V})$ is the tangent current to $T \wedge \theta$ associated to the same sequence. We also have $\kappa^V(T \wedge \theta) = \kappa^V(T) \smile \pi_0^*\{\theta_{|V}\}$ and $\kappa_j^V(T \wedge \theta) = \kappa_{j+q}^V(T) \smile \{\theta_{|V}\}$.

We consider now a case which is very useful in computing tangent classes.

Lemma 4.10. *Let X, V and the (p,p)-current T be as above. Assume that $p \leq l$ and that the tangential h-dimension of T along V is minimal, i.e. equal to $l-p$. Then $\kappa_{l-p}^V(T) = \{T\}_{|V}$ and $\kappa^V(T) = \pi_0^*(\{T\}_{|V})$. In particular, when V is a hypersurface of X and T has no mass on V, the above identities hold and we have moreover $\kappa^V(T)_{|H_\infty} = \{T\}_{|V}$.*

Proof. When V is a hypersurface, we have $\mathbb{P}(E) = V$ and V-conic currents without mass on V are pull-back by π_0 of currents on V. Moreover, π_0 defines an isomorphism between H_∞ and V. Therefore, the second assertion is a direct consequence of the first one. We prove the first assertion using the notation introduced above. For simplicity, assume that τ is smooth.

By Lemma 2.20, we have $\{T_\lambda\}_{|V} = \{T\}_{|V}$ for every λ. So if S is a tangent current to T along V, the class $\{S\}_{|V}$ is equal to $\{T\}_{|V}$. Lemma 3.4 implies that S is the pull-back by π_0 of the shadow S^h of S on V. Therefore, we have $\kappa^V(T) = \{S\} = \{\pi_0^*(S^h)\}$ and S^h belongs to the class $\{S\}_{|V} = \{T\}_{|V}$. On the other hand, by Lemma 3.8, S^h belongs to $\kappa_{l-p}(S) = \kappa_{l-p}^V(T)$. The lemma follows. □

The following result shows the upper semi-continuity for the maximal h-dimensional part of the tangent class when the current T varies.

Theorem 4.11. *Let X, V and W_1 be as in Theorem 4.6. Let T_n and T be positive closed (p,p)-currents on X with support in W_1 such that $T_n \to T$. Let s be the tangential h-dimension of T along V. Then*

1. *If r is an integer strictly larger than s, then $\kappa_r^V(T_n)$ converges to 0.*
2. *If κ_s is a limit class of the sequence $\kappa_s^V(T_n)$, then the classes κ_s and $\kappa_s^V(T) - \kappa_s$ are pseudo-effective.*

Proof. If T has positive mass on V, then the tangential h-dimension of T along V is maximal, i.e. equal to $k-p$. The theorem is clear. Assume now that T has no mass on V. We deduce that the mass of T_n on V tends to 0. So removing from T_n its restriction to V permits to assume that T_n has no mass on V for every n.

Denote by $\widehat{T}$ and $\widehat{T}_n$ the strict transforms of T and T_n with respect to the blow-up $\sigma : \widehat{X} \to X$ along V. Recall that we identify the hypersurface at infinity H_∞ of $\overline{E}$ with $\sigma_E^{-1}(H_\infty)$ and with $\mathbb{P}(E)$. So the restriction of a class κ to H_∞ or to $\sigma_E^{-1}(H_\infty)$ is denoted by $\kappa_{|\mathbb{P}(E)}$. By Lemma 4.7 and the last assertion of Lemma 4.10, we have

$$\kappa^V(T)_{|\mathbb{P}(E)} = \kappa^{\widehat{V}}(\widehat{T})_{|\mathbb{P}(E)} = \{\widehat{T}\}_{|\widehat{V}}$$

and a similar property for T_n. Extracting a subsequence we can assume that $\widehat{T}_n$ converges to a current $\widehat{T}'$. Write $\widehat{T}' = \widehat{T} + \widehat{R}$ where $\widehat{R}$ is the restriction of $\widehat{T}'$ to $\widehat{V}$. If $\{\widehat{R}\}'$ denotes the class of $\widehat{R}$ in $H^{2p}_{\widehat{V}}(\widehat{X}, \mathbb{C})$, we have

$$\lim_{n\to\infty} \kappa^V(T_n)_{|\mathbb{P}(E)} - \kappa^V(T)_{|\mathbb{P}(E)} = \{\widehat{R}\}'_{|\widehat{V}}.$$

We show that the h-dimension of $\widehat{R}$ is at most equal to s.

Assume that the h-dimension of $\widehat{R}$ is strictly larger than s. By Remarks 4.9, we can multiply T and T_n by a strictly positive closed form in order to assume that $\kappa^V(T) = 0$ and that $\widehat{R}$ is a vertical current. We then have $\lim \kappa^V(T_n)_{|\mathbb{P}(E)} = \{\widehat{R}\}'_{|\widehat{V}}$. On one hand the above limit is a pseudo-effective class. On the other hand, arguing as in Lemma 3.17, the class $\{\widehat{R}\}'_{|\widehat{V}}$ can be represented by a strictly negative constant times a linear subspace on a fiber of π. This is a contradiction.

So the h-dimension of $\widehat{R}$ is at most equal to s. We deduce from the above computation on $\lim \kappa^V(T_n)_{|\mathbb{P}(E)}$ that the h-dimension of this limit is at most equal to s. This gives us the first part of the theorem, see Lemma 3.15. Since the last limit is a pseudo-effective class, we also deduce that $\lim \kappa^V_s(T_n)$ is pseudo-effective. This implies that the class κ_s in the second part of the theorem is pseudo-effective. Finally, by Lemma 3.17, the class $-\kappa_s(\{\widehat{R}\}'_{|\widehat{V}})$ is pseudo-effective. This and the above computation imply that $\kappa^V_s(T) - \kappa_s$ is pseudo-effective and complete the proof of the theorem. □

Note that when T has no mass on V its total tangent class $\kappa^V(T)$ along V is determined by its restriction to the hypersurface at infinity $H_\infty \simeq \mathbb{P}(E)$. As above, we denote this class by $\kappa^V(T)_{|\mathbb{P}(E)}$. We identify both $\pi_0 : H_\infty \to V$ and $\sigma : \widehat{V} \to V$ with $\pi : \mathbb{P}(E) \to V$. The following proposition gives us a way to compute the tangent class of T along V. It is similar to Siu's point of view on the Lelong number at a point using the blow-up at this point, see Siu [37].

Proposition 4.12. *Let X, V and T be as in Theorem 4.11. Let $\widehat{T}$ be the strict transform of T with respect to the blow-up $\sigma : \widehat{X} \to X$ of X along V. Denote*

by $-h_{\mathbb{P}(E)}$ *the tautological class of* $\pi : \mathbb{P}(E) \to V$ *as above. Let* $e(T)$ *be the class in* $H^{2p-2}_c(\mathbb{P}(E), \mathbb{C})$ *defined in Lemma 3.14. Assume that* T *has no mass on* V*. Then*

$$\kappa^V(T)_{|\mathbb{P}(E)} = e(T) \smile h_{\mathbb{P}(E)} + \pi^*(\{T\}_{|V}).$$

Proof. For simplicity, we identify $\sigma_E^{-1}(H_\infty)$ with H_∞ and with $\mathbb{P}(E)$. We then have $\kappa^V(T)_{|\mathbb{P}(E)} = \kappa^{\widehat{V}}(\widehat{T})_{|\mathbb{P}(E)}$. By Lemma 4.10 applied to $\widehat{V}$, we have $\kappa^{\widehat{V}}(\widehat{T})_{|\mathbb{P}(E)} = \{\widehat{T}\}_{|\widehat{V}}$. Recall that $\{\widehat{T}\}$ is equal to the difference between $\sigma^*\{T\}$ and the canonical image $\widetilde{e}(T)$ of $e(T)$ in $H^{2p}_{\widehat{V}}(\widehat{X}, \mathbb{C})$. We also have $(\sigma^*\{T\})_{|\widehat{V}} = \pi^*(\{T\}_{|V})$ (this can be seen using a smooth form in $\{T\}$). Moreover,

$$\widetilde{e}(T)_{|\widehat{V}} = \big(\pi_0^*(e(T)) \smile [\widehat{V}]\big)_{|\widehat{V}} = e(T) \smile \{\widehat{V}\}_{|\widehat{V}} = -e(T) \smile h_{\mathbb{P}(E)}.$$

This implies the proposition. Note that $\{T\}_{|V} = 0$ when $p > \dim V$. □

The following result will be used to bound tangent classes and to show that some tangent classes vanish.

Proposition 4.13. *Let* X, V *and* T *be as above. Let* V' *be a submanifold of* V*. Let* s *denote the tangential h-dimension of* T *along* V*. Then the tangential h-dimension of* T *along* V' *is at most equal to* s*. Moreover, if* S *is a tangent current to* T *along* V*, we have* $\kappa_s^{V'}(T) \leq \kappa_s^{V'}(S)$*.*

If T has support in V, then $S = T$ and the proposition is clear. So we can assume that T has no mass on V. In particular, we have $s < k - p$.

Let τ be the smooth admissible map given in Lemma 4.2. Let $\sigma' : \widehat{X}' \to X$ be the blow-up of X along V' and $\sigma_{E'} : \widehat{E}' \to E'$ the blow-up along V' of the normal vector bundle E' to V' in X. Let $\widehat{T}'$ be the strict transform of T by $\sigma' : \widehat{X}' \to X$. Define $\widehat{V}' := \sigma'^{-1}(V')$ and we identify it with $\sigma_{E'}^{-1}(V')$ and also with $\mathbb{P}(E')$. Observe that in general τ is not admissible with respect to V'. We need the following lemma.

Lemma 4.14. *The map* τ *lifts to a bi-Lipschitz map* $\widehat{\tau}'$ *from a neighbourhood of* $\widehat{V}'$ *in* $\widehat{X}'$ *to a neighbourhood of* $\widehat{V}'$ *in* $\widehat{E}'$ *which is smooth outside* $\widehat{V}'$ *and preserves the hypersurface* $\widehat{V}'$*. Moreover, if* $\widetilde{T}' := \widehat{\tau}'_*(\widehat{T}')$*, we have* $\{\widehat{T}'\}_{|\widehat{V}'} = \{\widetilde{T}'\}_{|\widehat{V}'}$*.*

Proof. We have $\widehat{\tau}' = \sigma_{E'}^{-1} \circ \tau \circ \sigma'$ outside $\widehat{V}'$. We first show that this map extends to a bi-Lipschitz map. The map τ is described locally as in Definition 2.14 and in the proof of Lemma 4.2 where all functions involved are smooth. In order to simplify the notation, we will not use exactly the same coordinate system of $\mathbb{D}^k$ as above.

Let (y^1, y^2) denote a linear coordinates system on $\mathbb{D}^k$ where $y^1 = (y_1, \ldots, y_{l'}) := x^1$ but y^2 is obtained from (x^2, x^3) by an index permutation. We consider that the components of y^2 play an equivalent role. We can write in these coordinates

$$\tau(y) = \big(y^1 + y^2 b(y), y^2 c(y)\big) + O(\|y^2\|^2) \quad \text{as} \quad y^2 \to 0$$

where the functions involved in b, c and $O(\|y^2\|^2)$ are smooth and the determinant of the matrix $c(y)$ is equal to 1. In these coordinates we identify σ' with $\sigma_{E'}$.

We cover $\sigma'^{-1}(\mathbb{D}^k)$ with a finite number of equivalent charts and as above we will only work in one of them. The considered chart is denoted by $\widehat{D}'$ endowed with coordinates $w = (w^1, w^\#, w_k)$ with $w^1 := (w_1, \ldots, w_{l'})$, $w^\# := (w_{l'+1}, \ldots, w_{k-1})$, $|w_j| < 2$ such that

$$\sigma'(w) = (w^1, w_k w^\#, w_k) \quad \text{and} \quad \sigma_{E'}^{-1}(y) = \sigma'^{-1}(y) = (y^1, y_k^{-1} y^\#, y_k).$$

We deduce that

$$\widehat{\tau}'(w) = \big(w^1 + w_k \widetilde{b}(w) + O(|w_k|^2), \widetilde{c}^\#(w) + w_k^{-1} O(|w_k|^2), w_k \widetilde{c}_k(w) + O(|w_k|^2)\big)$$

where the functions involved in $\widetilde{b}$, $\widetilde{c}^\#$, $\widetilde{c}_k$ and $O(|w_k|^2)$ are smooth. The inverse of $\widehat{\tau}'$ satisfies a similar property. We see that $\widehat{\tau}'$ extends to a bi-Lipschitz map which is not identity on $\widehat{V}'$ in general. The hypersurface $\widehat{V}'$ is given by $w_k = 0$. So it is invariant under $\widehat{\tau}'$.

It remains to prove the last identity in the lemma. By Lemma 2.20, we only have to check that the restriction $\widetilde{\tau}$ of $\widehat{\tau}'$ to $\widehat{V}'$ acts trivially on $H^*_c(\widehat{V}', \mathbb{C})$. In local coordinates as above, we have $\widetilde{\tau}(w^1, w^\#) = (w^1, \widetilde{c}^\#(w^1, w^\#, 0))$. So it is induced by the differential of τ which is $\mathbb{C}$-linear at each point of V. We deduce that $\widetilde{\tau}$ is induced by a smooth self-map of the tautological line bundle $O_{\widehat{V}'}(-1)$ of $\widehat{V}'$ which sends $\mathbb{C}$-linearly fibers to fibers. It follows that $\widetilde{\tau}$ preserves the tautological class of $\widehat{V}'$. On the other hand, it preserves the fibers over V'. Hence, Leray's theorem implies that $\widetilde{\tau}$ acts trivially on $H^*_c(\widehat{V}', \mathbb{C})$. This completes the proof of the lemma. □

End of the proof of Proposition 4.13. Recall that T has no mass on V and $s < k-p$. By Lemma 4.10 applied to $\widehat{V}'$, we have $\kappa^{V'}(T)_{|\mathbb{P}(E')} = \{\widehat{T}'\}_{|\widehat{V}'}$. It follows from the last lemma that $\kappa^{V'}(T)_{|\mathbb{P}(E')} = \{\widetilde{T}'\}_{|\widehat{V}'}$. The map A_λ can be lifted to a holomorphic map $\widehat{A}_\lambda : \widehat{E}' \to \widehat{E}'$. Since this map depends continuously on λ, it acts trivially on cohomology with integer coefficients. Therefore, it acts trivially on de Rham cohomology. Thus, $\kappa^{V'}(T)_{|\mathbb{P}(E')} = \{\widehat{T}'_\lambda\}_{|\widehat{V}'}$ where $\widehat{T}'_\lambda := (\widehat{A}_\lambda)_*(\widetilde{T}')$. Define $T_\lambda := (A_\lambda)_* \tau_*(T)$ as above. We have $(\sigma_{E'})_*(\widehat{T}'_\lambda) = T_\lambda$. Let (λ_n) be a sequence such that T_{λ_n} converges to S. Then, $\widehat{T}'_{\lambda_n}$ converges outside $\widehat{V}'$ to the strict transform $\widehat{S}'$ of S by $\sigma_{E'}$. We show that any limit current of $\widehat{T}'_{\lambda_n}$ is equal to $\widehat{S}'$ plus a positive closed current supported by $\widehat{V}'$.

Let R be a smooth $(2k-2p)$-form with compact support in $\widehat{X}'$. We show that the family of $\langle \widehat{T}'_\lambda, R\rangle$ is bounded for λ large enough. Using a partition of unity, we reduce the problem to the case where R is supported by $\sigma'^{-1}(\mathbb{D}^k)$ as in Lemma 2.17. Since the considered currents have no mass on $\widehat{V}'$ and V', we have

$$\langle \widehat{T}'_\lambda, R\rangle = \langle T_\lambda, \sigma'_*(R)\rangle_{\mathbb{D}^k\setminus V'} = \big\langle T, \tau^*(A_\lambda)^* \sigma'_*(R)\big\rangle_{\mathbb{D}^k\setminus V'}.$$

Lemmas 2.6 and 2.17 imply that the family $\langle \widehat{T}'_\lambda, R\rangle$ is bounded. It follows that the family of currents $\widehat{T}'_\lambda$ is relatively compact.

By Lemma 2.16, if the component of bidegree $(k-p,k-p)$ of R vanishes, the above integral converges to 0. Therefore, the limit currents of $\widehat{T}'_\lambda$ are of bidegree (p,p). The same proposition shows that if R is a weakly positive $(k-p,k-p)$-form, then the limit values of $\langle \widehat{T}'_\lambda, R\rangle$ is positive. We conclude that the limit currents of $\widehat{T}'_{\lambda_n}$ are positive closed (p,p)-currents. Recall that these currents are equal to $\widehat{S}'$ outside $\widehat{V}'$. Let $\widehat{S}'+\widehat{S}''$ be such a limit current with $\widehat{S}''$ positive closed supported by $\widehat{V}'$. Denote by $\{\widehat{S}''\}'$ the class of $\widehat{S}''$ in $H^{2p}_{\widehat{V}}(\widehat{X},\mathbb{C})$.

We deduce from the above discussion that

$$\kappa^{V'}(T)_{|\mathbb{P}(E')} = \{\widehat{S}'\}_{|\widehat{V}'} + \{\widehat{S}''\}_{|\widehat{V}'} = \kappa^{V'}(S)_{|\mathbb{P}(E')} + \{\widehat{S}''\}'_{|\widehat{V}'}.$$

Let r denote the h-dimension of T along V'. If r is strictly larger than s, replacing T by $T\wedge\omega^r$ gives us identities similar to the last ones with $S=0$; this contradicts Lemma 3.17 applied to $\widehat{S}''$ and the fact that $\kappa_r^{V'}(T)_{|\mathbb{P}(E')}$ is pseudo-effective. So we have $r\leq s$. We also deduce that the h-dimension of $\widehat{S}''$ is at most equal to s. Then, using Lemmas 3.15 and 3.17 we obtain that

$$\kappa_s^{V'}(T) = \kappa_s^{V'}(S) + \kappa_s(\{\widehat{S}''\}'_{|\widehat{V}'}) \leq \kappa_s^{V'}(S).$$

The proposition follows. □

5 Density and intersection of currents

Let X be a Kähler manifold of dimension k as above. In this section we will introduce a notion of density associated to any finite family of positive closed currents such that the intersection of their supports is compact. The last condition is satisfied when X is already compact. We will study some basic properties of the density and compare it with the Lelong number. We also discuss a new notion of intersection of currents and compare it with classical notions.

Let T_j be a positive closed current of bidegree (p_j,p_j) on X with $1\leq j\leq m$. Assume that the intersection of their supports is compact. Define $\mathbb{T} := T_1\otimes\cdots\otimes T_m$. This is a positive closed (p,p)-current on X^m with $p:=p_1+\cdots+p_m$. Denote by Δ the diagonal of X^m, i.e. the set of points $(x,\ldots,x)$ with $x\in X$. It is canonically isomorphic to X. Then the intersection of supp$(\mathbb{T})$ with Δ is compact and $\mathbb{T}$ has no mass on Δ except when the T_j are measures which contain a same atom.

Denote by $\mathrm{Tan}(X)$, $\mathrm{Tan}(X^m)$ and $\mathrm{Tan}(\Delta)$ the tangent vector bundles of X, X^m and Δ respectively. Let $\mathbb{E}_m$ denote the normal bundle to Δ in X^m. The vectors which are tangent to the fibers of the natural projection $(x_1,\ldots,x_m)\mapsto(x_1,\ldots,x_{m-1})$ constitute a vector sub-bundle of $\mathrm{Tan}(X^m)$. Its restriction to Δ is

a complement of $\mathrm{Tan}(\Delta)$ in $\mathrm{Tan}(X^n)$. We see that $\mathbb{E}_m$ is canonically isomorphic to $\mathrm{Tan}(X)\oplus\cdots\oplus\mathrm{Tan}(X)$ ($m-1$ times). So the rank of $\mathbb{E}_m$ is equal to $(m-1)k$. Define

$$\kappa(T_1,\ldots,T_m) := \kappa^\Delta(\mathbb{T}).$$

This is a pseudo-effective cohomology class in $H^{2p}_c(\overline{\mathbb{E}}_m,\mathbb{C})$. Define also

$$\kappa_j(T_1,\ldots,T_m) := \kappa^\Delta_j(\mathbb{T}).$$

This is a cohomology class in $H^{2k-2j}_c(X,\mathbb{C})$.

Definition 5.1. The class $\kappa(T_1,\ldots,T_m)$ is called *the total density class*; the class $\kappa_j(T_1,\ldots,T_m)$ is *the density class of dimension* j and the h-dimension of $\kappa(T_1,\ldots,T_m)$ is *the density h-dimension* associated to $T_1,\ldots,T_m$. If $\mathbb{S}$ is a tangent current to $\mathbb{T}$ along Δ, we say that $\mathbb{S}$ is *a density current associated to* $T_1,\ldots,T_m$.

Observe that any permutation of $(x_1,\ldots,x_m)$ induces holomorphic automorphisms of $\mathbb{E}_m$ and of $\overline{\mathbb{E}}_m$ which leave invariant the fibers. So the action on $\overline{\mathbb{E}}_m$ preserves the tautological class $-h_{\overline{\mathbb{E}}_m}$. We then deduce from Leray's theorem that the action is in fact identity on $H^*_c(\overline{\mathbb{E}}_m,\mathbb{C})$. Therefore, κ and κ_j are symmetric in $T_1,\ldots,T_m$.

Example 5.2. If the currents T_j have locally continuous potentials, we can show that the current $\mathbb{T}$ admits a unique tangent current along Δ. This current vanishes when $p_1+\cdots+p_m>k$ and is equal to the pull-back of $T_1\wedge\ldots\wedge T_m$ otherwise, see also Proposition 5.10 below.

Lemma 5.3. *The density h-dimension associated to $T_1,\ldots,T_m$ is smaller or equal to the complex dimension $k-p_j$ of T_j for $1\leq j\leq m$.*

Proof. Let s denote the density h-dimension associated to $T_1,\ldots,T_m$. The lemma is clear if the density class vanishes. Suppose this is not the case. Then the class $\kappa_s(T_1,\ldots,T_m)$ is non-zero and pseudo-effective. We have

$$\kappa_s(T_1,\ldots,T_m)\smile\{\omega^s\}\neq 0.$$

The last class is also the shadow of $\kappa^\Delta(T_1\otimes\cdots\otimes T_m\wedge\Pi_j^*(\omega^s))$, where Π_j is the projection from X^m to j-th factor. We deduce that $T_1\otimes\cdots\otimes T_m\wedge\Pi_j^*(\omega^s)\neq 0$. It follows that $T_j\wedge\omega^s\neq 0$ and hence $s\leq k-p_j$. □

The following lemma shows that the notion of density generalizes the notion of tangent currents.

Lemma 5.4. *Let X,V and T be as in Section 4. Then $\kappa_j(T,[V])$ is equal to the canonical image of $\kappa^V_j(T)$ in $H^{2k-2j}_c(X)$.*

Proof. Observe that $[V] \otimes T$ can be identified with the pull-back of T by the canonical projection $\Pi : V \times X \to X$. The restriction of $\mathbb{E}_2$ to $V_\Delta := V \cap \Delta$ can be identified with the pull-back of the tangent vector bundle of X by the restriction $\Pi_{|V_\Delta}$ of Π to Δ_V. We denote it by F. The tangent currents to $[V] \otimes T$ along Δ can be identified with tangent currents to $\Pi^*(T)$ along V_Δ. The pull-back to V_Δ by Π of the tangent vector bundle of V is a sub-bundle of F that we denote by F'. The quotient F/F' can be identified with the normal vector bundle E to V in X if we identify V_Δ with V. Denote by $\rho : F \to F/F'$ the canonical projection.

We show that the tangent currents to $\Pi^*(T)$ along V_Δ are equal to the pull-back by ρ of the tangent currents to T along V. For this purpose, we will use local coordinates as in Section 4. We identify a chart of X with $\mathbb{D}^k = \mathbb{D}^l \times \mathbb{D}^{k-l}$ on which V is equal to $\mathbb{D}^l \times \{0\}$. Consider the natural coordinate system (x', y', y'') on the chart $\mathbb{D}^l \times \mathbb{D}^k$ of $V \times X$ where Δ_V is given by $\{y' = x', y'' = 0\}$ and $\Pi(x', y', y'') = (y', y'')$.

In order to identify V_Δ with V we use the coordinate system (x', z', y'') with $z' := y' - x'$. So V is identified with V_Δ and given by $\{z' = 0, y'' = 0\}$. The vector bundle F is then identified to $\mathbb{D}^l \times \mathbb{C}^k$ and F' is the intersection of $\mathbb{D}^l \times \mathbb{C}^k$ with the subspace $\{y'' = 0\}$. So it is equal to $\mathbb{D}^l \times \mathbb{C}^l \times \{0\}$. The vector bundles F/F' and E are identified to $\mathbb{D}^l \times \{0\} \times \mathbb{C}^{k-l}$. The map ρ is just the canonical projection $(x', z', y'') \mapsto (x', 0, y'')$.

The projection Π is given in these coordinates by $\Pi(x', z', y'') = (x' + z', 0, y'')$ where we identify the chart $\mathbb{D}^k \subset X$ with the polydisc $\mathbb{D}^l \times \{0\} \times \mathbb{D}^{k-l}$. We use the identity map as an admissible map associated to X and V. We also use the map $\tau(x', z', y'') := (x' + z', z', y'')$ for the pair $V \times X$ and V_Δ. The multiplication with λ on E and $\mathbb{E}_2$ are identified with the map $A_\lambda(x', z', y'') := (x', \lambda z', \lambda y'')$. We see that $\Pi \circ (A_\lambda \circ \tau)^{-1} = (A_\lambda)^{-1} \circ \rho$. It follows that $(A_\lambda)_* \tau_* \Pi^*(T) = \rho^*(A_\lambda)_*(T)$. Thus, if S is a tangent current to T along V then $\rho^*(S)$ is a tangent current to $\Pi^*(T)$ along V_Δ.

The map ρ induces a meromorphic map $\widetilde{\rho} : \overline{\mathbb{E}}_2 \to \overline{E}$. It is not difficult to show that if $-h_{\overline{E}}$ is the tautological class of $\overline{E}$ then $-\widetilde{\rho}^*(h_{\overline{E}})$ is the tautological class of $\overline{E}$. Finally, the uniqueness of the decomposition in Leray's theorem, implies that $\kappa_j(\widetilde{\rho}^*(S)) = \kappa_j(S)$. The lemma follows.

Note that the above construction gives an isomorphism between the set of tangent currents to T along V and the set of tangent currents to $[V] \otimes T$ along Δ. □

Note that in the last lemma the canonical morphism from $H^{2l-2j}_c(V, \mathbb{C})$ to $H^{2k-2j}_c(X, \mathbb{C})$ is not injective in general. However, the lemma still holds if we replace X by a small enough neighbourhood of V and in that case the corresponding morphism is injective.

Let T be a positive closed (p,p)-current on X. By Siu's theorem [37], the Lelong number $\nu(T, x)$ defines a function which is upper semi-continuous with

respect to the Zariski topology on X. In particular, if Y is an irreducible analytic set, then $\nu(T, \cdot)$ is constant on a dense Zariski open set of Y. We denote this constant by $\nu(T, Y)$. Moreover, also by Siu's theorem, there is a finite or countable family of irreducible analytic sets Y_j of dimension $k-p$ and constants $c_j > 0$ such that

$$T = \sum c_j [Y_j] + T'$$

where T' is a positive closed current such that for every $c > 0$ the level set $\{\nu(T, \cdot) \geq c\}$ is an analytic set of dimension $\leq k-p-1$. The following results give the relation between density of currents and Lelong numbers.

Lemma 5.5. *Let X and T_j be as above. Assume that T_1 is a measure, i.e. $p_1 = k$. Then the density h-dimension associated to $T_1, \ldots, T_m$ is 0 and we have*

$$\kappa_0(T_1, \ldots, T_m) = \big\langle T_1, \nu(T_2, \cdot) \ldots \nu(T_m, \cdot) \big\rangle.$$

In particular, the function $a \mapsto \kappa_0(\delta_a, T_2, \ldots, T_m)$ is upper semi-continuous with respect to the Zariski topology on X.

Proof. The first assertion is a consequence of Lemma 5.3. For the second assertion, since κ_0 is linear on each variable, we can disintegrate T_1 into Dirac masses and assume for simplicity that T_1 is the Dirac mass at a point a. In this case, we see that $\kappa_0(T_1, \ldots, T_m)$ is the Lelong number of $T_2 \otimes \cdots \otimes T_m$ at the point $(a, \ldots, a)$. It is not difficult to see that this Lelong number is equal to $\nu(T_2, a) \ldots \nu(T_m, a)$. The lemma follows. $\square$

Proposition 5.6. *Let X and T_j be as above. Assume that T_1 is the current of integration on an irreducible analytic set Y of dimension $k - p_1$. Then*

$$\kappa_{k-p_1}(T_1, \ldots, T_m) = \nu(T_2, Y) \ldots \nu(T_m, Y)\{Y\}.$$

Proof. Let $\mathbb{S}$ be a density current associated to $T_1, \ldots, T_m$. If $\kappa_{k-p_1}(T_1, \ldots, T_m)$ does not vanish, by Lemma 5.3, the h-dimension of $\mathbb{S}$ is equal to $k - p_1$. So the last class contains the shadow of $\mathbb{S}$ which is a positive closed (p_1, p_1)-current supported by Y. Therefore, it is equal to a constant c times $\{Y\}$. Of course, this property holds also when the considered class vanishes. We compute now the constant c.

We have

$$\kappa_0(T_1 \wedge \omega^{k-p_1}, T_2, \ldots, T_m) = \kappa_{k-p_1}(T_1, \ldots, T_m) \smile \{\omega^{k-p_1}\} = c\{T_1 \wedge \omega^{k-p_1}\}.$$

Since $T_1 \wedge \omega^{k-p_1}$ is a positive measure, by Lemma 5.5, we have

$$c\{T_1 \wedge \omega^{k-p_1}\} = \big\langle T_1 \wedge \omega^{k-p_1}, \nu(T_2, \cdot) \ldots \nu(T_m, \cdot) \big\rangle.$$

Since $T_1 = [Y]$, Siu's theorem mentioned above implies that the last integral is equal to $\nu(T_2, Y) \ldots \nu(T_m, Y)\{T_1 \wedge \omega^{k-p_1}\}$. The proposition follows. $\square$

Proposition 5.7. *Let X and T_j be as above. Assume that the set $\mathscr{E}$ of points x such that $\{\nu(T_2, x) > 0\}$ contains no analytic set of dimension $k - p_1$. Then the density h-dimension associated to $T_1, \ldots, T_m$ is strictly smaller than $k - p_1$.*

Proof. By Lemma 5.3, this dimension is at most equal to $k-p_1$. So it is enough to prove that $\kappa_{k-p_1}(T_1, \ldots, T_m) \smile \{\omega^{k-p_1}\} = 0$. Arguing as in the end of Proposition 5.6, we have

$$\kappa_{k-p_1}(T_1, \ldots, T_m) \smile \{\omega^{k-p_1}\} = \big\langle T_1 \wedge \omega^{k-p_1}, \nu(T_2, \cdot) \ldots \nu(T_m, \cdot)\big\rangle.$$

By hypothesis, $\mathscr{E}$ is a finite or countable union of analytic sets of dimension less that $k - p_1$. Therefore, the measure $T_1 \wedge \omega^{k-p_1}$ has no mass on $\mathscr{E} = \{\nu(T_2, \cdot) \neq 0\}$ and the last integral vanishes. The proposition follows. □

The following result is a direct consequence of Theorem 4.11.

Corollary 5.8. *Let X be $U_1, \ldots, U_m$ open subsets with relatively compact intersection in Kähler manifold X. Let $T_{j,n}$ and T_j be positive closed (p_j, p_j)-currents with support in U_j such that $T_{j,n} \to T_j$ as $n \to \infty$. Let s denote the density h-dimension associated to $T_1, \ldots, T_m$. Then $\kappa_j(T_{1,n}, \ldots, T_{m,n}) \to 0$ for $j > s$. Moreover, any limit class of the sequence $\kappa_s(T_{1,n}, \ldots, T_{m,n})$ is pseudo-effective and smaller or equal to $\kappa_s(T_1, \ldots, T_m)$.*

In what follows, we will introduce a new definition for the intersection of positive closed currents. We will give some basic properties needed in our dynamical setting. We believe that the theory has an independent interest and has to be developed. Let X and T_i be as above. We assume that $p_1 + \cdots + p_m \leq k$ which is a necessary condition to give a meaning to the intersection of the T_j.

Definition 5.9. Assume that the density h-dimension associated to $T_1, \ldots, T_m$ is minimal, i.e. equal to $k - p_1 - \cdots - p_m$. Assume that there is a unique density current $\mathbb{S}$ associated to $T_1, \ldots, T_m$. We define $T_1 \curlywedge \ldots \curlywedge T_m$ as the shadow of $\mathbb{S}$ with respect to the fibration $\pi : \overline{\mathbb{E}}_m \to \Delta$.

Observe that in this case $\mathbb{T} := T_1 \otimes \cdots \otimes T_m$ admits a unique tangent current $\mathbb{S}$ with respect to Δ and by Lemma 3.4, it is equal to the pull-back of the current $T_1 \curlywedge \ldots \curlywedge T_m$. We deduce that

$$\{T_1 \curlywedge \ldots \curlywedge T_m\} = \lim_{\lambda \to \infty} \{\mathbb{T}_\lambda\}_{|\Delta} = \{\mathbb{T}\}_{|\Delta},$$

where $\mathbb{T}_\lambda$ is defined as in Section 4 for $\mathbb{T}, \Delta$ instead of T, V. It follows that

$$\{T_1 \curlywedge \ldots \curlywedge T_m\} = \{T_1\} \smile \cdots \smile \{T_m\}.$$

The permutations of factors in X^m induce bi-holomorphic self-maps on $\overline{\mathbb{E}}_m$ which preserve the fibers of $\pi_0 : \overline{\mathbb{E}}_m \to X$. Since $S = \pi_0^*(T_1 \curlywedge \ldots \curlywedge T_m)$ these

bi-holomorphic maps also preserve $\mathbb{S}$. Hence, the wedge-product $T_1 \curlywedge \ldots \curlywedge T_m$ is symmetric with respect to $T_1, \ldots, T_m$.

Let T be a positive closed (p, p)-current and T' a positive closed $(1, 1)$-current with bounded local potentials. According to Bedford-Taylor [3], we can define the wedge-product $T' \wedge T$ by $T' \wedge T := dd^c(uT)$ when u is a local potential of T' which is a bounded psh function, see also Chern-Levine-Nirenberg [7], Demailly [10] and [26]. The definition does not depend on the choice of u and therefore extends to the global setting. This wedge-product gives a positive closed $(p+1, p+1)$-current.

When $T_1, \ldots, T_{m-1}$ are of bidegree $(1, 1)$ and have locally bounded potentials, we can define $T_1 \wedge \ldots \wedge T_m$ by induction. The following result compares this wedge-product with the definition given above. The proposition holds for currents with bounded local potentials but the proof in that case is more technical.

Proposition 5.10. *Let X and T_j be as above. Assume that $T_1, \ldots, T_{m-1}$ are of bidegree $(1, 1)$ with continuous local potentials. Then we have*

$$T_1 \wedge \ldots \wedge T_m = T_1 \curlywedge \ldots \curlywedge T_m.$$

Proof. Let U be a small open set in X that we identify with the unit polydisc in $\mathbb{C}^k$ using local coordinates $x = (x_1, \ldots, x_k)$. Then U^m is identified with the unit polydisc in $(\mathbb{C}^k)^m$ and we denote the canonical coordinate system by $(x^1, \ldots, x^m)$. We will use other coordinates $(y^1, \ldots, y^m)$ on U^m given by $y^m := x^m$ and $y^j := x^j - x^m$ for $1 \leq j \leq m - 1$. In these coordinates, $\Delta \cap U^m$ is given by the equations $y^1 = \cdots = y^{m-1} = 0$. The vector bundle $\mathbb{E}_m$ is identified over $\Delta \cap U^m$ with $(\mathbb{C}^k)^{m-1} \times U$ where the zero section, i.e. $\Delta \cap U^m$, is identified with $\{0\} \times U$.

In these coordinates, the application $\tau := \mathrm{id}$ is admissible. Write $T_j = dd^c u_j$ with u_j psh on U. Define a psh function $\widetilde{u}_j$ on U^m by $\widetilde{u}_j(y^1, \ldots, y^m) := u_j(y^j + y^m)$. The current $\widetilde{T}_j := dd^c \widetilde{u}_j$ is the pull-back of T_j to U^m by the projection from U^m onto the j-th factor. Using convolutions, we can approximate u_j uniformly by smooth psh functions. We see that $\mathbb{T} := T_1 \otimes \cdots \otimes T_m$ is equal to $\widetilde{T}_1 \wedge \ldots \wedge \widetilde{T}_m$ where $\widetilde{T}_m$ is the pull-back of T_m by the canonical projection $\Pi : (\mathbb{C}^k)^{m-1} \times U \to U$.

Let A_λ denote the multiplication by λ along the factor $(\mathbb{C}^k)^{m-1}$ in $(\mathbb{C}^k)^{m-1} \times U$. We have to show that $(A_\lambda)_*(\mathbb{T})$ converges to $\Pi^*(T_1 \wedge \ldots \wedge T_m)$ as $\lambda \to \infty$. We have

$$(A_\lambda)_*(\mathbb{T}) = dd^c(\widetilde{u}_1 \circ A_\lambda^{-1}) \wedge \ldots \wedge dd^c(\widetilde{u}_{m-1} \circ A_\lambda^{-1}) \wedge \widetilde{T}_m.$$

Since $\widetilde{u}_j \circ (A_\lambda)^{-1}$ converges locally uniformly to $u_j \circ \Pi$, the last wedge-product converges to $\Pi^*(T_1 \wedge \ldots \wedge T_m)$. This completes the proof of the proposition.

Note that the same proof also works for currents of higher bidegree which admit local continuous potentials. □

We will also need the following lemma in the dynamical setting.

Lemma 5.11. *Let X, U_j, T_j be as above. Let $1 < l < m$ be an integer. Assume that $U := U_{l+1} \cap \ldots \cap U_m$ is relatively compact in X and that T_j is of bidegree*

$(1,1)$ *and has local continuous potentials in a neighbourhood of* $\overline{U}$ *for* $1 \leq j \leq l$. *Assume also that* $T_{l+1} \curlywedge \ldots \curlywedge T_m$ *exists. Then* $T_1 \curlywedge \ldots \curlywedge T_m$ *exists and we have*

$$T_1 \curlywedge \ldots \curlywedge T_m = T_1 \wedge \ldots \wedge T_l \wedge (T_{l+1} \curlywedge \ldots \curlywedge T_m).$$

Proof. Using the similar notation as in Proposition 5.10, we have

$$(A_\lambda)_*(\mathbb{T}) = dd^c(\widetilde{u}_1 \circ A_\lambda^{-1}) \wedge \ldots \wedge dd^c(\widetilde{u}_l \circ A_\lambda^{-1}) \wedge (A_\lambda)_*(T_{l+1} \otimes \cdots \otimes T_m).$$

Observe that $\widetilde{u}_j \circ A_\lambda^{-1}$ converges locally uniformly to $u_j \circ \Pi$ and by hypotheses $(A_\lambda)_*(T_{l+1} \otimes \cdots \otimes T_m)$ converges to $\Pi^*(T_{l+1} \curlywedge \ldots \curlywedge T_m)$. We deduce that the right hand side of the last identity converges to

$$\Pi^*\big(T_1 \wedge \ldots \wedge T_l \wedge (T_{l+1} \curlywedge \ldots \curlywedge T_m)\big).$$

The lemma follows. $\square$

Remark 5.12. The hypothesis on the supports of the currents T_j can be refined. We can extend the notion of tangent current along a manifold V to currents which satisfy suitable regularity near the points of $V \setminus K$ for some compact subset K of V. It would be also useful to compare the above notion with the notion introduced in [20, 21]. We believe that the new notion extends the previous one and is valid in a more general setting.

Remark 5.13. A subset $\mathscr{E}$ of $\mathbb{R}^+$ is said to be *of density zero* if the Lebesgue measure of $E \cap [0, n]$ is equal to $o(n)$ when $n \to \infty$. Let T_λ be as in Section 4. Assume that for some set $\mathscr{E}$ of zero density the limit of T_λ exists when $\lambda \to \infty$ and $|\lambda| \notin \mathscr{E}$. We then say that the limit is *the essential tangent current to* T *along* V. We can consider a notion of intersection of currents $T_1, \ldots, T_m$ by assuming the existence of the essential tangent current to $T_1 \otimes \cdots \otimes T_m$ along Δ. We can also use another measure instead of the Lebesgue measure on $\mathbb{R}^+$ or take some average before considering the limit.

The following example illustrates an advantage of the above notion of intersection.

Example 5.14. Let π_1 denote the canonical projection from $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor. Let a be a point in $\mathbb{P}^1$ and ν a positive measure on $\mathbb{P}^1$ having no mass at a. Consider two positive closed currents $T_1 := \pi_1^*(\delta_a)$ and $T_2 := \pi_1^*(\nu)$. It is not difficult to check that $T_1 \curlywedge T_2 = 0$. If the local potentials of ν are equal to $-\infty$ at a, then the local potentials of T_2 are equal to $-\infty$ on the support of T_1. In this case, we cannot define $T_1 \wedge T_2$ in the classical sense.

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